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Decision Support

Loss-averse preferences and portfolio choices: An extension☆

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ABSTRACT

In this paper we generalise existing models of loss-averse preferences. This extension clarifies the impact of stochastic changes in risk on the optimal degree of risk taking. Our more general model highlights an intuitive link between the literature on loss-averse behaviours and the notions of prudence and temperance recently introduced in the literature. We also stress the link between our approach and the use of VaR and CVaR as risk measures.

1. Introduction

In the standard expected utility model (E-U) it is now well known that beneficial changes in the returns of a risky asset do not necessarily induce more risk taking by a risk averse decision maker (see, e.g., Fishburn and Porter, 1976, for the original contribution and Eeckhoudt and Gollier, 2013, for a survey).

In the E-U model with risk averse investors the utility function is assumed to be increasing and concave *everywhere*. This assumption has been challenged at least since Fishburn (1977) who developed an alternative model of choice under risk, the so-called "$\alpha - t$" model. In Fishburn (1977) the decision maker is concerned by a target level of wealth (denoted t) above which risk neutrality prevails. Below the target the decision maker pays attention to the extent of the failure to reach the target and this concern is expressed by the use of a *specific* concave power function, the exponent of which is denoted α (hence the "$\alpha - t$" terminology).

A little later, Kahneman and Tversky (1979) suggested to express loss aversion through a piecewise-linear utility function that is globally concave. In this model the loss of 1 euro below the target is more painful (in absolute value) than the pleasure attached to the gain of 1 euro above the target.

Other models expressing loss aversion with a linear utility above the target and a *specific* concave utility below the target have been suggested (for a short and useful review see Jarrow and Zhao, 2006, especially Section 2).

In this paper we postulate a more general version of loss aversion in which the utility function below the target is concave (so that we no longer refer to a *specific* power utility function) while linearity prevails above the target. Besides at the target level, the slope of the concave part equals that of the linear component (so that in the spirit of loss aversion models the loss of 1 euro below the target is more painful than a similar loss above the target).

As shown in Section 4, with such a model of loss aversion, stochastic improvements in the return of risky assets induce more risk taking under much less restrictive assumptions about the utility function than those prevailing under the general E-U model. Besides, and maybe more importantly, these less restrictive assumptions are linked to the notions of prudence (and temperance) that are relatively recent in the literature. As a result we can provide a link between the literature on loss aversion and new concepts for the analysis of risky choices.

While our results are first expressed in terms of classical stochastic dominance relationships between random variables, we next develop the special cases of (a) two risks with equal first moments that fulfill Rothschild and Stiglitz's (1970) definition of "increasing risk", and (b) two risks with equal means and variances that represent an increase in downside risk according to Menezes, Geiss, and Tressler (1980). An interesting result obtained in Section 4 is that prudence and temperance jointly constitute a sufficient condition for a

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loss-averse investor to decrease his optimal exposure to a risky asset whose downside risk increases. When modifications in risk involve a tail risk parameter, such as Value at Risk (VaR) or Conditional Value at Risk (CVaR), the reaction of a loss-averse agent can still be predicted unequivocally under mild requirements, that we describe in Section 5. As a consequence of the relationship between stochastic dominance and risk measures proved in Ma and Wong (2010), the proposed model of downside loss aversion emerges as a decision-theoretic foundation for VaR and CVaR in the asset management industry.

Our paper is organized as follows. In Section 2 we present notation and assumptions. In Section 3 we briefly review the original contribution of Fishburn and Porter (1976) and we introduce the model of loss aversion used all along the paper. Section 4 contains our main results on (first, second and n-th order) Stochastic Dominance, followed by some special cases of stochastic changes in risk. In Section 5 we provide a relationship between our results and VaR and CVaR. Some concluding remarks are summarized in the final section.

2. Notation and assumptions

We consider a standard portfolio problem with one safe and one risky asset. The decision maker is endowed with a sure wealth $w_0 > 0$ that can be invested in a safe asset with return $i \geq 0$ or a risky asset with random return X so that the final wealth W is a random variable defined by

$$W = (w_0 - a)(1 + i) + a(1 + X) \tag{1}$$

where a is the amount of money invested in the risky asset. To simplify further notations, (1) can also be written

$$W = w^i + a(X - i)$$

where $w^i \triangleq w_0(1 + i)$ is the end of period wealth of a portfolio fully invested in the riskless security.

The problem of the agent is to choose a value $a^* \in [0; w_0]$ that maximizes the expected utility

$$V(a) = E[u(W)] \tag{2}$$

of his final wealth, where u is a Von Neumann–Morgenstern utility function. The value of a^* expresses the optimal exposure to risk X.

In the following, X is supposed to be valued in a finite interval $[\underline{x}, \overline{x}]$, with $\underline{x} \geq -1$, and to have an arbitrary, right-continuous cumulative distribution function F. As pointed out by Gollier (2001), for the problem to make sense the excess return variable $(X - i)$ must alternate in sign, otherwise a^* would be either $+\infty$ or $-\infty$.

Economists usually consider specific changes in F such as stochastic dominance orders. Here, we will use and recall the definitions of n-th order Stochastic Dominances.

Let $\mathcal{F}$ be the set of right-continuous distribution functions of random variables valued in $[\underline{x}, \overline{x}]$. For each $F \in \mathcal{F}$, let $F^1 = F$ and recursively define:

$$F^n(x) = \int_{\underline{x}}^{x} F^{n-1}(y)dy$$

for all $n \in \{2, 3, \ldots\}$, $x \in \mathbb{R}$. A distribution function $G \in \mathcal{F}$ is said to dominate F by n-th degree stochastic dominance (here denoted by $F \preceq_n G$) if:

$$G^n(x) \leq F^n(x) \quad \text{for all } x \in [\underline{x}, \overline{x}]$$

A first degree stochastic improvement from F to G is thus characterized by the following condition:

$$F \preceq_1 G \Leftrightarrow G(x) \leq F(x) \quad \text{for all } x \in [\underline{x}, \overline{x}]$$

which may be interpreted as an increase in the probability of observing large values of X. A second degree stochastic improvement from F to G:

$$F \preceq_2 G \Leftrightarrow G^2(x) \leq F^2(x) \quad \text{for all } x \in [\underline{x}, \overline{x}]$$

makes the random variable X both "larger" and "less variable" (Shaked & Shanthikumar, 2007).

When the first $(n - 1)$ moments of F and G are equal, nth-order stochastic dominance coincides with Ekern's (1980) notion of nth-degree risk reduction. For instance, G is a reduction in second degree risk over F if $F \preceq_2 G$ and $E_F(X) = E_G(X)$. This is what Rothschild and Stiglitz (1970) have termed a *mean-preserving reduction in risk*. Improvements in third and fourth degree risk have subsequently received an intuitive interpretation in terms of downside risk and outer risk reductions, respectively (Menezes et al., 1980; Menezes & Wang, 2005). All these characterisations, as well as their extensions to higher orders, are synthesized by Eeckhoudt and Schlesinger (2006), who establish that stochastic improvements of general order n play a crucial role in connection with risk apportionment decisions.

To describe the effect of stochastic changes in risk on risk taking, we will make use of the following concepts related to the agent's utility function. The Arrow–Pratt coefficient of absolute risk aversion is defined as

$$ARA(w) = -\frac{u''(w)}{u'(w)},$$

while $RRA(w) = wRA(w)$ is the agent's coefficient of relative risk aversion, a measure of the relative risk premium associated with a multiplicative risk.

The index:

$$P(w) = -\frac{u'''(w)}{u''(w)}$$

is called the degree of absolute prudence (a measure of the convexity of u') and $RP(w) = wP(w)$ is relative prudence[1]. It is well-known that prudence is a necessary condition for decreasing absolute risk aversion, or DARA (see, e.g., Gollier, 2001, pp. 24–25, for a formal definition and related implications).

3. Brief review of the literature and statement of our model

Although the simple portfolio problem is a classical topic in the economics of uncertainty, the impact on the optimal investment mix of a deterioration in the risky asset distribution (according to some degree of stochastic dominance) has challenged economists since the publication of Fishburn and Porter's (1976) paper on the subject.

3.1. A quick look at Fishburn and Porter (1976)

In a market consisting of one risky asset and one riskless security, Fishburn and Porter (1976) characterize the optimal investment mix of an agent whose utility function is twice continuously differentiable and strictly increasing. The first two derivatives of expected utility (2) are given by:

$$V'(a) = \int_{\underline{x}}^{\overline{x}} u'[w^i + a(x - i)](x - i)dF(x)$$

$$V''(a) = \int_{\underline{x}}^{\overline{x}} u''[w^i + a(x - i)](x - i)^2 dF(x).$$

If $u'' > 0$ (risk seeking) or $u'' = 0$ (risk neutral, u linear), the optimal risk exposure is either $a^* = 0$ or $a^* = w_0$. In order to rule out uninteresting solutions, Fishburn and Porter (1976) restrict their attention to the case of a globally risk averse utility function ($u'' < 0$). Then $V''(a) < 0$ and there is a unique solution $a^* \in [0, w_0]$ that maximizes (2). The

[1] The concept of prudence will be analysed in more detail in Section 4. For an economic interpretation of the benchmark values of 1 for relative risk aversion and 2 for relative prudence, we refer the reader to Eeckhoudt, Etner, and Schroyen (2009).