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Short Communication

A note on two-stage network DEA model: Frontier projection and duality

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ABSTRACT

In Chen, Cook, Kao, and Zhu (2013), it is demonstrated, as a network DEA pitfall, that while the multiplier and envelopment DEA models are dual models and equivalent under the standard DEA, such is not necessarily true for the two types of network DEA models in deriving divisional efficiency scores and frontier projections. As a reaction to this work, we demonstrate that the duality in the standard DEA naturally migrates to the two-stage network DEA. Formulas are developed to obtain frontier projections and divisional efficiency scores using a DEA model's and its dual solutions. The case of Taiwanese non-life insurance companies is revisited using the newly developed approach.

1. Introduction

Data envelopment analysis (DEA) is an approach for measuring relative efficiency or calculating a composite benchmarking index when multiple performance measures (or inputs and outputs) are present in decision making units (DMUs). In recent years, a significant research has been done on DMUs with internal structures. See Castelli et al. (2010), Cook, Liang, and Zhu (2010), and Kao (2014) for excellent reviews on this field. Among a wide variety of internal structures studied, one basic and popular internal structure is called a (basic) two-stage network process where outputs from the first stage (referred to as *intermediate measures*) become the inputs to the second stage.

While several approaches have been suggested to assess the efficiency of two-stage network processes in the literature, one simple approach (referred to as the standard DEA approach) is to deal with two individual stages as independent DMUs and then measure their efficiencies separately. For example, Seiford and Zhu (1999) evaluate the performance of US commercial banks under an independent two-stage process structure, where the first stage referred to as profitability uses labor and assets to produce profits and revenue, and subsequently the second stage referred to as marketability transforms the profits and revenue into market value, returns and earnings per share. Sexton and Lewis (2003) evaluate the performance of 30 teams in two Major League Baseball leagues, whose operations are seen as a two-stage process of the front-office and on-field operations. Chilingerian and Sherman (2004) examine the performance of physician care by considering it as a two-stage process; the first stage is the manager-controlled production where the hospital managers set up and manage the assets of the hospitals, and the second stage is the physician-controlled production where the physicians decide how and when to utilize these assets to provide the medical service to the patients. All of these studies apply the standard DEA models, such as CCR (Charnes, Cooper, & Rhodes, 1978) or BCC (Banker, Charnes, & Cooper, 1984) models, to measure the individual stages' and system's efficiency scores, separately and independently.

Although the standard DEA approach discussed above are simple and convenient to use, they may give rise to potential conflicts between the two stages arising from the intermediate measures. For instance, while the second stage may need to reduce its inputs (i.e., intermediate measures) in order to attain efficiency, such an adjustment would imply a reduction in the first stage outputs, thereby deteriorating that stage's efficiency (Cook et al., 2010). To address this conflict issue, various DEA models have been suggested including notably Kao and Hwang (2008), which develop what is called centralized model in Liang, Cook, and Zhu (2008). The key features of their model are that the overall DMU efficiency is decomposed into a product of two stages' efficiency scores, and that the intermediate measures are given the same weights no matter whether they are considered inputs or outputs. Liang et al. (2008) also develop two leader-follower models when either the first or second stage is assumed to be the "leader" in performance evaluation. Kao and Hwang (2008) claim that their model is more reliable in measuring the efficiencies and consequently is capable of identifying the causes of

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inefficiency more accurately than the standard DEA approach, but they do not provide a way to obtain frontier projections for inefficient DMUs, which are required for performance benchmarking.

Chen, Cook, and Zhu (2010) are the first to note that the approach of Kao and Hwang (2008) or the centralized model of Liang et al. (2008) does not produce frontier projections for inefficient DMUs using the efficiency scores. They propose to solve an additional envelopment type of DEA model that is equivalent (or dual) to the centralized model to determine optimal values for the intermediate measures. Furthermore, in Chen, Cook, Kao, and Zhu (2013), it is demonstrated, as a network DEA pitfall, that while the multiplier and envelopment DEA models are dual models and equivalent under the standard DEA, such is not necessarily true for the two types of network DEA models when deriving information for stage or divisional efficiency scores and frontier projections.

As a reaction to these works, the current study demonstrates that the duality in the standard DEA naturally migrates to the two-stage network DEA. We develop formulas that use the linear program dual variables to calculate system and stage efficiency scores and frontier projections. As such, our proposed approach improves and simplifies the procedure outlined in Chen et al. (2010). The rest of the paper unfolds as follows. Section 2 presents a generic two-stage network system, Kao and Hwang's model for measuring its efficiency, and Chen et al.'s (2010) procedure for deriving the DEA frontier. Section 3 is devoted to the development of simple formulas for frontier projections for inefficient DMUs, followed by a discussion of the relationship between frontier projections and efficiency scores under the two-stage network structure in Section 4. Section 5, as a numerical illustration, revisits the performance evaluation of the Taiwanese non-life insurance companies studied in Kao and Hwang (2008). Section 6 concludes.

2. Two-stage network system and efficiency measurement

Consider the generic two-stage process as shown in Fig. 1, for each of a set of n DMUs. A conventional description of the process in the literature is as follows: Each DMU j ($j = 1, 2, \ldots, n$) has m inputs x_{ij} ($i = 1, 2, \ldots, m$) to the first stage, and D outputs z_{dj}, ($d = 1, 2, \ldots, D$) from that stage. These D outputs then become the inputs to the second stage and will be referred to as intermediate measures. The outputs from the second stage are y_{rj}, ($r = 1, 2, \ldots, s$). In this paper, however, we modify the above convention slightly as follows: x_{ij} are regarded as the *system inputs*, not the divisional inputs to the first stage, and y_{rj} are regarded as the *system outputs*, not the divisional outputs from the second stage. On the other hand, the intermediate measures are considered as the *divisional outputs* from the first stage and the *divisional inputs* to the second stage. The reason for this modification will be later clarified in Section 4, where it is shown that the approach of Kao and Hwang (2008) implicitly supports this modified convention.

The input-oriented two-stage network DEA of Kao and Hwang (2008) or the centralized model of Liang et al. (2008) for measuring the efficiency of DMU 0 is given as follows:

$$\text{(P)}\quad \begin{array}{ll} \max & uy_o \\ \text{s.t.} & wZ - vX \le 0, \\ & uY - wZ \le 0, \\ & vx_0 = 1, \\ & v,\ u,\ w \ge 0, \end{array}$$

where $X = (x_{ij}) \in R^{m\times n}$, $Z = (z_{dj}) \in R^{D\times n}$, and $Y = (y_{rj}) \in R^{s\times n}$ are data matrices of inputs, intermediate measures, and outputs, respectively, and v, w, and u are optimization variables of appropriate dimensions representing optimal multipliers on factors. Notice that the same weights (w) are assigned to the intermediate measures no matter whether they are used as inputs or outputs. The optimal objective value (u^*y_0) to model (P) is the system efficiency score (denoted by θ^*), and the first and second stages' efficiency scores are determined by $\theta_1^* = \frac{w^*z_0}{v^*x_0} = w^*z_0$ and $\theta_2^* = \frac{u^*y_0}{w^*z_0}$, respectively, where $*$ denotes an optimal solution to (P). Note also that the system efficiency score is decomposed into a product of the two stages' efficiency scores; $\theta^* = \theta_1^* \cdot \theta_2^* = u^*y_0$.

While Kao and Hwang (2008) and Liang et al. (2008) do not discuss how to obtain frontier projections for inefficient DMUs, Chen et al. (2010) point out that the usual procedure of adjusting the inputs or outputs by the efficiency scores, as in the standard DEA approach, does not necessarily yield a frontier projection. The same argument is also presented in Chen et al. (2013) as a network DEA pitfall. Chen et al. (2010) suggest that the following additional envelopment type of DEA model should be solved to obtain frontier projections for inefficient DMUs:

$$\text{(D1)}\quad \begin{array}{ll} \min & \theta \\ \text{s.t.} & X\lambda \le \theta x_o, \\ & Z\lambda \ge \tilde{z}_0, \\ & Z\mu \le \tilde{z}_0 \\ & Y\mu \ge y_0, \\ & \lambda,\ \mu \ge 0, \end{array}$$

where θ, λ, μ, and $\tilde{z}_0$ are optimization variables of appropriate dimensions. Once model (D1) is solved, a frontier projection for DMU 0 is given by $(\theta^* x_0,\ \tilde{z}_0^*,\ y_0)$.

3. Frontier projections

Although Chen et al.'s (2010) procedure outlined in Section 2 is valid, it necessitates solving an additional linear program to obtain frontier projections, which is not the case with the standard DEA. In the standard DEA, duality holds between the multiplier model and the envelopment model, and this makes frontier projections readily obtainable from primal-dual optimal solution pairs. Chen et al.'s (2010) procedure may imply such duality is not readily usable under the two-stage DEA, as also indicated in Chen et al. (2013) as a network DEA pitfall. In this section, however, we show that the duality in the standard DEA naturally migrates to the two-stage network DEA, and develop simple formulas that use primal-dual optimal solution pairs to model (P) to readily determine frontier projections, thereby simplifying and improving the procedure of Chen et al. (2010).

When we solve model (P) using a usual linear program software for DMU 0 under evaluation, we get an optimal primal solution $(v^*,\ w^*,\ u^*)$ as well as a dual optimal solution $(\lambda^*,\ \mu^*,\ \theta^*)$ to the following:

$$\text{(D)}\quad \begin{array}{ll} \min & \theta \\ \text{s.t.} & X\lambda \le \theta x_o, \\ & Z\lambda \ge Z\mu, \\ & Y\mu \ge y_0, \\ & \lambda,\ \mu \ge 0. \end{array}$$

Then, a frontier projection for DMU 0 can be determined as follows:

Frontier projection: $(\tilde{x}_0,\ \tilde{z}_0,\ \tilde{y}_0) = (X\lambda^*,\ \tilde{z}_0,\ Y\mu^*)$ where $\tilde{z}_0$ is any choice such that $Z\mu^* \le \tilde{z}_0 \le Z\lambda^*$. An easy choice for $\tilde{z}_0$ would be $Z\mu^*$ or $Z\lambda^*$.

In the following proposition, we prove that the above formula truly yields a frontier projection by showing that it attains a system efficiency score of unity and its introduction (to the DMU set) does not move the current frontier.

Proposition 1. *$(\tilde{x}_0,\ \tilde{z}_0,\ \tilde{y}_0) = (X\lambda^*,\ \tilde{z}_0,\ Y\mu^*)$ is a frontier projection.*

Proof. We first show that $(\tilde{x}_0,\ \tilde{z}_0,\ \tilde{y}_0)$ has a system efficiency score of unity. The efficiency of $(\tilde{x}_0,\ \tilde{z}_0,\ \tilde{y}_0)$ is evaluated by solving the LP