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Original Article

A new class defined by subordination for γ-spirallike functions

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Abstract In this paper we shall introduce and study some subordination results for the class of γ-spirallike univalent functions defined by convolution.

MSC: 30C45; 30C80; 30D30,

1. Introduction

Let $\mathcal{A}$ denote the class of the functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic and univalent in the open unit disc $U = \{z : |z| < 1\}$. Let $f \in \mathcal{A}$ be given by (1.1) and g be given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k. \tag{1.2}$$

Definition 1. Let a function f defined by (1.1) and g defined by (1.2), the Hadamard product (or convolution) $(f*g)$ is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.3}$$

Definition 2 ([1]). A function $f(z) \in \mathcal{A}$ is in $S^{\gamma}(\alpha)$, the class of γ-spirallike functions of order α $(0 \le \alpha < 1, |\gamma| < \frac{\pi}{2})$, if and only if

$$\mathrm{Re}\left\{ e^{i\gamma} \frac{zf'(z)}{f(z)} \right\} > \alpha \cos\gamma \quad (z \in U). \tag{1.4}$$

We note that $S^{\gamma}(0) = S^{\gamma}$ (the class of γ -spirallike functions) was introduced by Spacek [2] (also see [3]) and $S^0(0) = S^*$ (see

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Silverman [4]). Further, a function $f(z)$ belonging to $\mathcal{A}$ is said to be in the class $C^{\gamma}(\alpha)$ if and only if

$$\mathrm{Re}\left\{e^{i\gamma}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > \alpha\cos\gamma \quad (z\in U). \tag{1.5}$$

We note that $C^{\gamma}(0)=C^{\gamma}$, the class of functions $f(z)$, for which $zf'(z)$ is γ -spirallike in U introduced by Robertson [5] and the class $C^{\gamma}(\alpha)$ was introduced and studied by Chichra [6] and Sizuk [7]. From (1.4) and (1.5) it follows that:

$$f(z)\in C^{\gamma}(\alpha) \iff zf'(z)\in \mathcal{S}^{\gamma}(\alpha).$$

Definition 3 [8]. (Subordination Principle). For two functions $f(z)$ and $F(z)$, analytic in U, we say that $f(z)$ is subordinate to $F(z)$, written symbolically as follows:

$$f \prec F \text{ in } U \text{ or } f(z)\prec F(z)(z\in U),$$

if there exists a Schwarz function $\omega(z)$, which is analytic in U with

$$\omega(0)=0 \text{ and } |\omega(z)|<1(z\in U)$$

such that

$$f(z)=F(\omega(z))(z\in U).$$

Indeed it is known that

$$f(z)\prec F(z)(z\in U)\Rightarrow f(0)=F(0) \text{ and } f(U)\subset F(U).$$

In particular, if the function $F(z)$ is univalent in U, we have the following equivalence

$$f(z)\prec F(z)(z\in U) \iff f(0)=F(0) \text{ and } f(U)\subset F(U).$$

Dziok and Srivastava [9] defined a linear operator $H_{q,s}(\alpha_1):\mathcal{A}\longrightarrow\mathcal{A}$ as follows:

$$H_{q,s}(\alpha_1)f(z)=z+\sum_{k=2}^{\infty}\Gamma_k(\alpha_1)a_k z^k, \tag{1.6}$$

where

$$\Gamma_k(\alpha_1)=\frac{(\alpha_1)_{k-1}\ldots(\alpha_q)_{k-1}}{(\beta_1)_{k-1}\ldots.(\beta_s)_{k-1}}\cdot\frac{1}{(1)_{k-1}}\;(k\geq 2).$$

The linear operator $H_{q,s}(\alpha_1)$ includes (as its special cases) various other linear operators for example Carlson and Shaffer [10], Ruscheweyh [11] and others.

For fixed $A, B(-1\leq B<A\leq 1)$, $0\leq\lambda<1$ and $|\gamma|<\frac{\pi}{2}$, we define the subclass $S_{\lambda}^{\gamma}(f,g;A,B)$ of $\mathcal{A}$ consisting of functions f of the form (1.1) and functions g is given by (1.2), with $b_k\geq 0$ as follows:

$$e^{i\gamma}\frac{zF_{\lambda}'(f,g)(z)}{F_{\lambda}(f,g)(z)}\prec\cos\gamma\frac{1+Az}{1+Bz}+i\sin\gamma, \tag{1.7}$$

where

$$zF_{\lambda}'(f,g)(z)=z(f*g)'(z)+\lambda z^2(f*g)''(z),$$

and

$$F_{\lambda}(f,g)(z)=(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)$$

From (1.7) and the definition of subordination, we obtain

$$e^{i\gamma}\frac{zF_{\lambda}'(f,g)(z)}{F_{\lambda}(f,g)(z)}=\cos\gamma\frac{1+A\omega(z)}{1+B\omega(z)}+i\sin\gamma,\ \omega(z)\in\Omega$$

and hence

$$\left|\frac{\frac{zF_{\lambda}'(f,g)(z)}{F_{\lambda}(f,g)(z)}-1}{B\frac{zF_{\lambda}'(f,g)(z)}{F_{\lambda}(f,g)(z)}-[B+(A-B)\cos\gamma e^{-i\gamma}]}\right|<1. \tag{1.8}$$

We note that:

(i) Putting $g(z)=\frac{z}{1-z}$ and $\lambda=0$, we have $S_0^{\gamma}(f,\frac{z}{1-z};A,B)=S^{\gamma}(A,B)$ (see Aouf [12], with $\alpha=0$);
(ii) Putting $g(z)=\frac{z}{1-z}$, $\lambda=0$, $A=1$ and $B=-1$, we have $S_0^{\gamma}(f,\frac{z}{1-z};1,-1)=S^{\gamma}$ (see Spacek [2]);
(iii) Putting $g(z)=\frac{z}{1-z}$, $\lambda=0$, $A=1-2\alpha(0\leq\alpha<1)$ and $B=-1$, we have $S_0^{\gamma}(f,\frac{z}{1-z};1-2\alpha,-1)=S^{\gamma}(\alpha)$ (see Libera [1] and Kwon and Owa [13]);
(iv) Putting $g(z)=\frac{z}{1-z}$, $\lambda=1$, $A=1$ and $B=-1$, we have $S_1^{\gamma}(f,\frac{z}{1-z};1,-1)=C^{\gamma}$ (see Robertson [5]);
(v) Putting $g(z)=\frac{z}{1-z}$, $\lambda=1$, $A=1-2\alpha(0\leq\alpha<1)$ and $B=-1$, we have $S_1^{\gamma}(f,\frac{z}{1-z};1-2\alpha,-1)=C^{\gamma}(\alpha)$ (see Chichra [6] and Sizuk [7]);
(vi) Putting $g(z)=\frac{z}{1-z}$, $\lambda=0$, $A=(1-2\alpha)\beta$ and $B=-\beta(0\leq\alpha<1, 0<\beta\leq 1)$, we have $S^{\gamma}(f,\frac{z}{1-z};(1-2\alpha)\beta,-\beta)=S^{\gamma}(\alpha,\beta)$;
(vii) Putting $g(z)=\frac{z}{1-z}$, $\lambda=1$, $A=(1-2\alpha)\beta$ and $B=-\beta(0\leq\alpha<1, 0<\beta\leq 1)$, we have $S^{\gamma}(f,\frac{z}{1-z};(1-2\alpha)\beta,-\beta)=C^{\gamma}(\alpha,\beta)$.

Also we note that:

(i) Putting $g(z)=z+\sum_{k=2}^{\infty}\Gamma_k(\alpha_1)z^k$, we have $S_{\lambda}^{\gamma}(f,z+\sum_{k=2}^{\infty}\Gamma_k(\alpha_1)z^k;A,B)=S_{\lambda}^{\gamma}(f,H_{q,s}(\alpha_1);A,B)$

$$=\left\{f\in\mathcal{A}:e^{i\gamma}\frac{z(H_{q,s}(\alpha_1)f(z))'+\lambda z^2(H_{q,s}(\alpha_1)f(z))''}{(1-\lambda)(H_{q,s}(\alpha_1)f(z))+\lambda z(H_{q,s}(\alpha_1)f(z))'}\prec\cos\gamma\frac{1+Az}{1+Bz}+i\sin\gamma,z\in U\right\},$$

where $H_{q,s}(\alpha_1)$ is given by (1.6);
(ii) Putting $g(z)=z+\sum_{k=2}^{\infty}(\frac{1+\ell+\delta(k-1)}{1+\ell})^m z^k$, where $\delta\geq 0$; $\ell\geq 0$ and $m\in\mathbb{N}_0$, we have $S_{\lambda}^{\gamma}(f,z+\sum_{k=2}^{\infty}(\frac{1+\ell+\delta(k-1)}{1+\ell})^m z^k;A,B)=S_{\lambda}^{\gamma}(f,I_{\delta,\ell}^m;A,B)$

$$=\left\{f\in\mathcal{A}:e^{i\gamma}\frac{z(I_{\delta,\ell}^m f(z))'+\lambda z^2(I_{\delta,\ell}^m f(z))''}{(1-\lambda)(I_{\delta,\ell}^m f(z))+\lambda z(I_{\delta,\ell}^m(\alpha_1)f(z))'}\prec\cos\gamma\frac{1+Az}{1+Bz}+i\sin\gamma,z\in U\right\},$$

where $I_{\delta,\ell}^m$ is Catas operator (see [14]);
(iii) Putting $g(z)=z+\sum_{k=2}^{\infty}\binom{k+\eta-1}{\eta}z^k$, where $\eta>-1$, we have $S_{\lambda}^{\gamma}(f,z+\sum_{k=2}^{\infty}\binom{k+\eta-1}{\eta}z^k;A,B)=S_{\lambda}^{\gamma}(f,D^{\eta};A,B)$

$$=\left\{f\in\mathcal{A}:e^{i\gamma}\frac{z(D^{\eta}f(z))'+\lambda z^2(D^{\eta}f(z))''}{(1-\lambda)(D^{\eta}f(z))+\lambda z(D^{\eta}f(z))'}\prec\cos\gamma\frac{1+Az}{1+Bz}+i\sin\gamma,z\in U\right\},$$

where D^{η} is Ruscheweyh derivative [11] defined by

$$D^{\eta}f(z)=\frac{z(z^{\eta-1}f(z))^{\eta}}{\eta!}=\frac{z}{(1-z)^{\eta+1}}*f(z);$$

(iv) Putting $g(z)=z+\sum_{k=2}^{\infty}k^n z^k$, where $n\in\mathbb{N}_0$, we have $S_{\lambda}^{\gamma}(f,z+\sum_{k=2}^{\infty}k^n z^k;A,B)=S_{\lambda}^{\gamma}(f,D^n;A,B)$

$$=\left\{f\in\mathcal{A}:e^{i\gamma}\frac{z(D^n f(z))'+\lambda z^2(D^n f(z))''}{(1-\lambda)(D^n f(z))+\lambda z(D^n f(z))'}\right.$$