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Original Article

On statistical approximation properties of q-Baskakov–Szász–Stancu operators

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Abstract In the present paper, we consider Stancu type generalization of Baskakov–Szász operators based on the q-integers and obtain statistical and weighted statistical approximation properties of these operators. Rates of statistical convergence by means of the modulus of continuity and the Lipschitz type maximal function are also established for operators.

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1. Introduction

In the recent years several operators of summation-integral type have been proposed and their approximation properties have been discussed. In the present paper our aim is to investigate statistical approximation properties of a Stancu type q-Baskakov–Szász operators. Firstly, Baskakov–Szász operators based on q-integers was introduced by Gupta [1] and some approximation results were established. The q-Baskakov–Szász operators are defined as follows:

$$\mathcal{D}_n^q(f,x)=[n]_q\sum_{k=0}^{\infty}p_{n,k}^q(x)\int_0^{q/1-q^n}q^{-k-1}s_{n,k}^q(t)f(t\,q^{-k})\mathrm{d}_qt, \quad (1.1)$$

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where $x \in [0, \infty)$ and

$$p^q_{n,k}(x) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{k(k-1)/2} \frac{x^k}{(1+x)_q^{n+k}}, \tag{1.2}$$

and

$$s^q_{n,k}(t) = E(-[n]_q t) \frac{([n]_q t)^k}{[k]_q!}. \tag{1.3}$$

In case $q = 1$, the above operators reduce to the Baskakov–Szász operators [2].

Later, Mishra and Sharma [3] introduced a new Stancu type generalization of q-Baskakov–Szász operators, which is defined as

$$\mathfrak{D}_n^{(\alpha,\beta)}(f; q; x) = [n]_q \sum_{k=0}^{\infty} p^q_{n,k}(x) \int_0^{q/1-q^n} q^{-k-1} s^q_{n,k}(t) f \times \left(\frac{[n]_q t q^{-k} + \alpha}{[n]_q + \beta} \right) \mathrm{d}_q t, \tag{1.4}$$

where $p^q_{n,k}(x)$ and $s^q_{n,k}(t)$ are Baskakov and Szász basis function respectively, defined as above. The operators $\mathcal{D}_n^{(\alpha,\beta)}(f; q; x)$ in (1.4) are called q-Baskakov–Szász-Stancu operators. For $\alpha = 0$, $\beta = 0$ the operators (1.4) reduce to the operators (1.1).

In the recent years several researchers have worked on Stancu type generalization of different operators and they have obtained various approximation properties. We mention some of important papers as [4–8].

Before proceeding further, we recall certain notations of q-calculus as follows. Such notations can be found in [9,10]. We consider q as a real number satisfying $0 < q < 1$.

For

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & q \neq 1, \\ n, & q = 1, \end{cases}$$

and

$$[n]_q! = \begin{cases} [n]_q [n-1]_q [n-2]_q \ldots [1]_q, & n = 1, 2, \ldots, \\ 1, & n = 0. \end{cases}$$

Then for $q > 0$ and integers $n, k, k \geq n \geq 0$, we have

$$[n+1]_q = 1 + q[n]_q \quad \text{and} \quad [n]_q + q^n [k-n]_q = [k]_q.$$

We observe that

$$(1+x)_q^n = (-x; q)_n = \begin{cases} (1+x)(1+qx)(1+q^2x)\cdots(1+q^{n-1}x), & n = 1, 2, \ldots, \\ 1, & n = 0. \end{cases}$$

Also, for any real number α, we have

$$(1+x)_q^\alpha = \frac{(1+x)_q^\infty}{(1+q^\alpha x)_q^\infty}.$$

In special case, when α is a whole number, this definition coincides with the above definition.

The q-Jackson integral and q-improper integral defined as

$$\int_0^a f(x) d_q x = (1-q) a \sum_{n=0}^{\infty} f(aq^n) q^n$$

and

$$\int_0^{\infty/A} f(x) d_q x = (1-q) a \sum_{n=0}^{\infty} f\left(\frac{q^n}{A}\right) \frac{q^n}{A},$$

provided sum converges absolutely.

The q-analogues of the exponential function e^x (see [10]), used here is defined as

$$E_q(z) = \prod_{j=0}^{\infty} (1 + (1-q) q^j z) = \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{z^k}{[k]_q!} = (1 + (1-q) z)_q^\infty, \ |q| < 1,$$

where $(1-x)_q^\infty = \prod_{j=0}^{\infty} (1 - q^j x)$.

2. Moment estimates

Lemma 1. *[1] The following hold:*

1. $\mathcal{D}_n(1, q; x) = 1$,
2. $\mathcal{D}_n(t, q; x) = x + \frac{q}{[n]_q}$,
3. $\mathcal{D}_n(t^2, q; x) = \left(1 + \frac{1}{q[n]_q}\right) x^2 + \frac{x}{[n]_q}(1 + q(q+2)) + \frac{q^2(1+q)}{[n]_q^2}$.

Lemma 2 ([3]). *The following hold:*

1. $\mathfrak{D}_n^{(\alpha,\beta)}(1; q; x) = 1$,
2. $\mathfrak{D}_n^{(\alpha,\beta)}(t; q; x) = \frac{[n]_q x + q + \alpha}{[n]_q + \beta}$,
3. $\mathfrak{D}_n^{(\alpha,\beta)}(t^2; q; x) = \left(\frac{[n]_q (q[n]_q + 1)}{q([n]_q + \beta)^2} \right) x^2 + \left(\frac{(1 + q(q+2))[n]_q + 2\alpha [n]_q}{([n]_q + \beta)^2} \right) x + \frac{q^2(1+q) + 2q\alpha + \alpha^2}{([n]_q + \beta)^2}$.

3. Korovkin type statistical approximation properties

The idea of statistical convergence goes back to the first edition (published in Warsaw in 1935) of the monograph of Zygmund [11]. Formerly the concept of statistical convergence was introduced by Steinhaus [12] and Fast [13] and later reintroduced by Schoenberg [14]. Statistical convergence, while introduced over nearly 50 years ago, has only recently become an area of active research. Different mathematicians studied properties of statistical convergence and applied this concept in various areas.

In approximation theory, the concept of statistical convergence was used in the year 2002 by Gadjiev and Orhan [15]. They proved the Bohman–Korovkin type approximation theorem for statistical convergence. It was shown that the statistical versions are stronger than the classical ones.

Korovkin type approximation theory also has many useful connections, other than classical approximation theory, in other branches of mathematics (see Altomare and Campiti in [16]).

Let us recall the concept of a limit of a sequence extended to a statistical limit by using the natural density δ of a set K of positive integers:

$$\delta(K) = \lim_n n^{-1} \{\text{the number } k \leq n \text{ such that } k \in K\}$$

whenever the limit exists (see [17], p. 407). So, the sequence $x = (x_k)$ is said to be statistically convergent to a number L, meaning that for every $\epsilon > 0$,

$$\delta\{k : |x_k - L| \geq \epsilon\} = 0$$