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Original Article

Some fixed point theorems for *G*-isotone mappings in partially ordered metric spaces

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Abstract Fixed point theorems for *G*-isotone mappings, which extend some recent results for mixed monotone and isotone mappings in partially ordered metric spaces are proved. Moreover, the equivalence between unidimensional and multidimensional fixed point theorems is investigated.

MSC: Primary 47H10; 54H25

1. Introduction

Following paper [1], the problem of existence of a fixed point for contraction type mappings in partially ordered metric spaces has been considered a lot (see, e.g., [2–22] and the related references therein). Some fixed point theorems were proved in these papers and they are usually applied in discussing the existence and uniqueness of solution to matrix equations, periodic boundary value problems and nonlinear integral equations.

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Recently, Roldán et al. [17] introduced the notion of coincidence point between mappings in any number of variables and extended several special notions of, so called, coupled, tripled, quadrupled and multidimensional fixed/coincidence points appeared in the literature see, for example, [3], [8], [14], [15], respectively. Results in [17] also extend some fixed points ones in the framework of partially ordered complete metric spaces. In order to guarantee the existence of coincidence point the authors of [17] constructed some Cauchy sequences using the properties of mixed monotone mappings and contractive conditions. The idea was used in a lot of paper (see, e.g., [16], [18], [19]). To prove that more than one sequences are simultaneously Cauchy's, seems not so easy. It is also known that the fixed point problems for isotone mappings are easier than that of mixed monotone mappings. Wang [21] obtained some multidimensional fixed point theorems for isotone mappings and extended some of the results in coupled, tripled, quadrupled and multidimensional fixed/coincidence points for mixed monotone and non-decreasing mappings in

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partially ordered complete metric spaces. She also gave a simple and unified approach to coupled, tripled, quadrupled and multidimensional fixed point theorems for mixed monotone mappings.

Motivated and inspired by the above results, we obtain some new fixed point theorems for G-isotone mappings and investigate the equivalence between unidimensional and multidimensional fixed point theorems.

2. Preliminaries

Let $n \in \mathbb{N}$, X be a non-empty set and X^n be the Cartesian product of n copies of X. For brevity, $g(x)$, $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$, $(z_1, z_2, \ldots, z_n)$, $(v_1, v_2, \ldots, v_n)$ and $(x_0^1, x_0^2, \ldots, x_0^n)$ will be denoted by gx, X, Y, Z, V and X_0, respectively.

Let $\{A, B\}$ be a partition of the set $\Lambda_n = \{1, 2, \ldots, n\}$, that is, $A \cup B = \Lambda_n$ and $A \cap B = \emptyset$, $\Omega_{A,B} = \{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq A$ and $\sigma(B) \subseteq B\}$ and $\Omega'_{A,B} = \{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq B$ and $\sigma(B) \subseteq A\}$. Let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be n mappings from Λ_n into itself. If $(X, \preceq)$ is a partially ordered space, $y, v \in X$ and $i \in \Lambda_n$, we use the next notation from [17]:

$$y \preceq_i v \Leftrightarrow \begin{cases} y \preceq v, & \text{if } i \in A, \\ y \succeq v, & \text{if } i \in B. \end{cases} \tag{1}$$

If elements x, y of a partially ordered set $(X, \preceq)$ are comparable (i.e. $x \preceq y$ or $y \preceq x$ holds) we will write $x \approx y$. The product space X^n is endowed with the following natural partial order: for Y, $V \in X^n$

$$Y \preceq_n V \Longleftrightarrow y_i \preceq_i v_i, i \in \Lambda_n. \tag{2}$$

The mapping $\rho_n : X^n \times X^n \to [0, +\infty)$, given by:

$$\rho_n(X, Y) = \max_{1 \le i \le n} d(x_i, y_i), \tag{3}$$

defines a metric on X^n. We denote Γ the set of all continuous and strictly increasing functions φ: $[0, \infty) \to [0, \infty)$, and Ψ the set of all functions ψ: $[0, \infty) \to [0, \infty)$, such that $\lim_{t \to r} \psi(t) > 0$ for every $r > 0$ and $\psi(t) = 0 \Longleftrightarrow t = 0$.

Definition 2.1 ([11]). A triple $(X, d, \preceq)$ is called an ordered metric space if (X, d) is a metric space and $(X, \preceq)$ is a partially ordered set.

Definition 2.2 ([17]). Let $g : X \to X$ be a mapping. If $(X, d, \preceq)$ is an ordered metric space, then X is said to have the sequential g-monotone property if it satisfies the following properties:

(i) If $(x_m)_{m \in \mathbb{N}}$ is a non-decreasing sequence and $\lim_{m\to\infty} x_m = x$, then $gx_m \preceq gx$ for all $m \in \mathbb{N}$.
(ii) If $(y_m)_{m \in \mathbb{N}}$ is a non-increasing sequence and $\lim_{m\to\infty} y_m = y$, then $gy_m \succeq gy$ for all $m \in \mathbb{N}$.

If g is the identity mapping, then X is said to have the sequential monotone property (see [17]) and $(X, d, \preceq)$ is said to be regular (see [22]).

Definition 2.3 ([16]). Let $F : X^n \to X$ and $g : X \to X$ be two mappings. A point $(x_1, x_2, \ldots, x_n) \in X^n$ is a Y-coincidence point of F and g if

$$F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)}) = gx_i$$

for $i \in \Lambda_n$. If g is the identity mapping on X, then $(x_1, x_2, \ldots, x_n) \in X^n$ is called a Y-fixed point of the mapping F.

Definition 2.4 ([19]). Let $(X, d, \preceq)$ be an ordered metric space. The mappings $F : X^n \to X$ and $g : X \to X$ are said to be O-compatible if, for all sequences $\{x_m^1\}_{m\ge0}, \{x_m^2\}_{m\ge0}, \ldots, \{x_m^n\}_{m\ge0} \subset X$ such that $\{gx_m^1\}_{m\ge0}, \{gx_m^2\}_{m\ge0}, \ldots, \{gx_m^n\}_{m\ge0}$ are monotone and the following limit exists: for all i,

$$\lim_{m\to\infty} F(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)}) = \lim_{m\to\infty} gx_m^i \in X,$$

we have

$$\lim_{m\to\infty} d(gF(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)}), F(gx_m^{\sigma_i(1)}, gx_m^{\sigma_i(2)}, \ldots, gx_m^{\sigma_i(n)})) = 0$$

for all i.

Definition 2.5 ([17]). Let $(X, \preceq)$ be a partially ordered space, and $F : X^n \to X$ and $g : X \to X$ be two mappings. It is said that F has the mixed g-monotone property if F is g-monotone nondecreasing in arguments with indices in A and g-monotone nonincreasing in arguments with indices in B, i.e., for all $x_1, x_2, \ldots, x_n, y, z \in X$ and each $i \in \{1, \ldots, n\}$,

$$gy \preceq gz \Rightarrow F(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) \preceq_i F(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n).$$

Definition 2.6 ([20]). Let $(X^n, \preceq)$ be a partially ordered set, and T and G self-mappings of X^n. It is said that T is a G-isotone mapping if, for any $Y_1, Y_2 \in X^n$

$$G(Y_1) \preceq_n G(Y_2) \Rightarrow T(Y_1) \preceq_n T(Y_2).$$

Definition 2.7 ([20]). An element $Y \in X^n$ is called a coincidence point of the mappings $T : X^n \to X^n$ and $G : X^n \to X^n$ if $T(Y) = G(Y)$. Furthermore, if $T(Y) = G(Y) = Y$, then is said that Y is a common fixed point of T and G.

Remark 2.8. Note that if $G = I_{X^n}$ in Definitions 2.6 and 2.7, then T is an isotone mapping and Y is a fixed point of T (see [21]).

Definition 2.9. $\mathcal{C}$ a family functions $f : [0, \infty)^2 \to R$ is called C-class if it is continuous and satisfies following axioms:

(1) $f(s, t) \le s$;
(2) $f(s, t) = s$ implies that either $s = 0$ or $t = 0$;

for all $s, t \in [0, \infty)$.

Example 2.10. The following functions $f : [0, \infty)^2 \to R$ are elements of $\mathcal{C}$. For each $s, t \in [0, \infty)$,

(1) $f(s, t) = ks$, $0 < k < 1$, $f(s, t) = s \Rightarrow s = 0$;
(2) $f(s, t) = s - t$, $f(s, t) = s \Rightarrow t = 0$;
(3) $f(s, t) = \frac{s-t}{1+t}$, $f(s, t) = s \Rightarrow t = 0$;
(4) $f(s, t) = \frac{s}{1+t}$, $f(s, t) = s \Rightarrow s = 0$ *or* $t = 0$;
(5) $f(s, t) = \log \frac{t+a^s}{1+t}$, $a > 1$, $f(s, t) = s \Rightarrow s = 0$ *or* $t = 0$;
(6) $f(s, t) = (s + l)(\frac{1}{1+t}) - l$, $l > 1$, $f(s, t) = s \Rightarrow t = 0$;
(7) $f(s, t) = s \log_{a+t} a$, $a > 1$, $f(s, t) = s \Rightarrow s = 0$ *or* $t = 0$.

Remark 2.11. Functions of C-class is a natural generalization for Banach contraction, as that can see in above example number (1).