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Original Article

On commutativity of rings with generalized derivations

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Abstract Let R be a prime ring, extended centroid C, Utumi quotient ring U, and $m, n \geq 1$ are fixed positive integers, F a generalized derivation associated with a nonzero derivation d of R. We study the case when one of the following holds: (i) $F(x)\circ_m d(y) = (x \circ y)^n$ and (ii) $(F(x) \circ d(y))^m = (x \circ y)^n$, for all x, y in some appropriate subset of R. We also examine the case where R is a semiprime ring.

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1. Introduction

In all that follows, unless specifically stated otherwise, R will be an associative ring, $Z(R)$ the center of R, Q its Martindale quotient ring and U its Utumi quotient ring. The center of U, denoted by C, is called the extended centroid of R (we refer the reader to [1], for the definitions and related properties of these objects). For any $x, y \in R$, the symbol $[x, y]$ and $x \circ y$ stands for the commutator $xy - yx$ and anti-commutator $xy + yx$, respectively. Given $x, y \in R$, we set $x\circ_0 y = x$, $x\circ_1 y = x \circ y = xy + yx$, and inductively $x\circ_m y = (x\circ_{m-1} y) \circ y$ for $m > 1$. Recall that a ring R is prime if $xRy = \{0\}$ implies either $x = 0$ or $y = 0$, and R is semiprime if $xRx = \{0\}$ implies $x = 0$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + yd(x)$ holds for all $x, y \in R$. In particular d is an inner derivation induced by an element $q \in R$, if $d(x) = [q, x]$ holds for all $x \in R$. If R is a ring and $S \subseteq R$, a mapping $f : R \to R$ is called strong commutativity-preserving (scp) on S if $[f(x), f(y)] = [x, y]$ for all $x, y \in S$.

Many results in the literature indicate that the global structure of a ring R is often tightly connected to the behavior of additive mappings defined on R. Derivations with certain properties investigated in various papers (see for Refs. [2–4]). Starting from these results, many authors studied generalized derivations in the context of prime and semiprime rings. By a generalized inner derivation on R, one usually means an additive mapping $F : R \to R$ if $F(x) = ax + xb$ for fixed $a, b \in R$. For such a mapping F, it is easy to see that $F(xy) = F(x)y +$

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$x[y, b] = F(x)y + xI_b(y)$. This observation leads to the definition given in [5]: an additive mapping $F : R \to R$ is called generalized derivation associated with a derivation d if $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. Familiar examples of generalized derivations are derivations and generalized inner derivations, and the latter includes left multipliers (i.e., an additive mapping $f(xy) = f(x)y$ for all $x, y \in R$). Since the sum of two generalized derivations is a generalized derivation, every map of the form $F(x) = cx + d(x)$ is a generalized derivation, where c is a fixed element of R and d is a derivation of R.

In [6], Lee extended the definition of a generalized derivation as follows: by a generalized derivation we mean an additive mapping $F : I \to U$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in I$, where I is a dense right ideal of R and d is a derivation from I into U. Moreover, Lee also proved that every generalized derivation can be uniquely extended to a generalized derivation on U, and thus all generalized derivations of R will be implicitly assumed to be defined on the derivation F on dense right ideal of R can be uniquely extended to U and assumes the form $F(x) = ax + d(x)$ for some $a \in U$ and a derivation d on U (see Theorem 3, in [6]). More related results about generalized derivations can be found [7,8].

During the past few decades, there has been an ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain specific types of derivations (see [9], where further references can be found). In [9], Ashraf and Rehman prove that if R is a prime ring, I is a nonzero ideal of R and d is a nonzero derivation of R such that $d(x \circ y) = x \circ y$ for all $x, y \in I$, then R is commutative. In [10], Argaç and Inceboz generalized the above result as following: Let R be a prime ring, I a nonzero ideal of R and n a fixed positive integer, if R admits a nonzero derivation d with the property $(d(x \circ y))^n = x \circ y$ for all $x, y \in I$, then R is commutative. In [8, Theorem 2.3], Quadri et al., discussed the commutativity of prime rings with generalized derivations. More precisely, Quadri et al., prove that if R is a prime ring, I a nonzero ideal of R and F a generalized derivation associated with a nonzero derivation d such that $F(x \circ y) = x \circ y$ for all $x, y \in I$, then R is commutative. In 2012 Huang [11], generalized the result obtained by Quadri et al., and he proved that if R is a prime ring, I a nonzero ideal of R, n a fixed positive integer and F a generalized derivation associated with a nonzero derivation d such that $(F(x \circ y))^n = x \circ y$ for all $x, y \in I$, then R is commutative.

In 1994 Bell and Daif [12], initiated the study of strong commutativity-preserving maps and prove that a nonzero right ideal I of a semiprime ring is central if R admits a derivation which is scp on I. In 2002 Ashraf and Rehman [9], prove that if R is a 2-torsion free prime ring, I is a nonzero ideal of R and d is a nonzero derivation of R such that $d(x) \circ d(y) = x \circ y$ for all $x, y \in I$, then R is commutative. The present paper is motivated by the previous results and we here generalized the result obtained in [9,11]. Moreover, we continue this line of investigation by examining what happens if a ring R satisfies the identity.

(i) $(F(x) \circ d(y))^m = (x \circ y)^n$ for all $x, y \in I$.
(ii) $F(x) \circ_m d(y) = (x \circ y)^n$ for all $x, y \in I$.

We obtain some analogous results for semiprime rings in the case $I = R$.

Explicitly we shall prove the following theorems:

Theorem 1.1. *Let R be a prime ring, I a nonzero ideal of R, and m, n are fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $(F(x) \circ d(y))^m = (x \circ y)^n$ for all $x, y \in I$, then R is commutative.*

Theorem 1.2. *Let R be a prime ring, I a nonzero ideal of R, and m, n are fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $F(x) \circ_m d(y) = (x \circ y)^n$ for all $x, y \in I$, then R is commutative.*

Theorem 1.3. *Let R be a semiprime ring, U the left Utumi quotient ring of R, and m, n are fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $(F(x) \circ d(y))^m = (x \circ y)^n$ for all $x, y \in R$, then R is commutative.*

Theorem 1.4. *Let R be a semiprime ring, U the left Utumi quotient ring of R, and m, n are fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $F(x) \circ_m d(y) = (x \circ y)^n$ for all $x, y \in R$, then R is commutative.*

2. The results in prime rings

We will make frequent use of the following result due to Kharchenko [13] (see also [14]):

Let R be a prime ring, d a nonzero derivation of R and I a nonzero two sided ideal of R. Let $f(x_1, \dots x_n, d(x_1, \dots x_n))$ be a differential identity in I, that is

$$f(r_1, \dots r_n, d(r_1), \dots, d(r_n)) = 0 \text{ for all } r_1, \dots, r_n \in I.$$

One of the following holds:

(1) Either d is an inner derivation in Q, the Martindale quotient ring of R, in the sense that there exists $q \in Q$ such that $d = ad(q)$ and $d(x) = ad(q)(x) = [q, x]$, for all $x \in R$, and I satisfies the generalized polynomial identity

$$f(r_1, \dots, r_n, [q, r_1], \dots, [q, r_n]) = 0;$$

(2) or, I satisfies the generalized polynomial identity

$$f(x_1, \dots, x_n, y_1, \dots, y_n) = 0.$$

Theorem 1.1. *Let R be a prime ring, I a nonzero ideal of R, and m, n are fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $(F(x) \circ d(y))^m = (x \circ y)^n$ for all $x, y \in I$, then R is commutative.*

Proof. If $F = 0$, then $(x \circ y)^n = 0$ for all $x, y \in I$, which can be rewritten as $(xy + yx)^n = 0$. If $char(R) \neq 2$, then $(2x^2)^n = 0$ for all $x \in I$. This is a contradiction by Xu [15]. If $char(R) = 2$, then $(xy + yx)^n = 0 = [x, y]^n$ for all $s, y \in I$. Thus by Herstein [16, Theorem 2], we have $I \subseteq Z(R)$, and so R is commutative by Mayne [17]. Hence, onward we will assume that $F \neq 0$ and $(F(x) \circ d(y))^m = (x \circ y)^n$ for all $x, y \in I$. By Lee [6, Theorem 3], every generalized derivation of R will be implicitly assumed to be defined on dense right ideal of R can be uniquely extended to U and assumes the form $F(x) = ax + d(x)$ for some $a \in U$ and a derivation d on U. Therefore, I satisfies the polynomial identity

$$((ax + d(x)) \circ d(y))^m = (x \circ y)^n \text{ for all } x, y \in I.$$

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