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**Original Article** 

# On projection-invariant submodules of QTAG-modules



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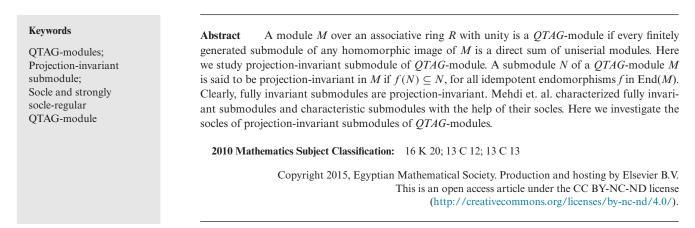
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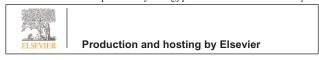
#### 1. Introduction and preliminaries

All the rings *R* considered here are associative with unity and modules *M* are unital *QTAG*-modules. An element  $x \in M$ is uniform, if *xR* is a non-zero uniform (hence uniserial) module and for any *R*-module *M* with a unique composition se-

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ries, d(M) denotes its composition length. For a uniform element  $x \in M$ , e(x) = d(xR) and  $H_M(x) = \sup \{d(\frac{yR}{xR}) | y \in M, x \in yR$  and y uniform} are the exponent and height of x in M, respectively.  $H_k(M)$  denotes the submodule of M generated by the elements of height at least k and  $H^k(M)$  is the submodule of M generated by the elements of exponents at most k. M is h-divisible if  $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$  and it is h-reduced if it does not contain any h-divisible submodule. In other words it is free from the elements of infinite height. A QTAG-module M is said to be separable, if  $M^1 = 0$ . A family  $\mathcal{N}$  of submodules of M is called a nice system in M if.

- (i)  $0 \in \mathcal{N};$
- (ii) If  $\{N_i\}_{i\in I}$  is any subset of  $\mathcal{N}$ , then  $\Sigma_I N_i \in \mathcal{N}$ ;
- (iii) Given any  $N \in \mathcal{N}$  and any countable subset X of M, there exists  $K \in \mathcal{N}$  containing  $N \cup X$ , such that K/N is countably generated [1].

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A *h*-reduced *QTAG*-module *M* is called totally projective if it has a nice system. A submodule  $B \subseteq M$  is a basic submodule of *M*, if *B* is *h*-pure in *M*,  $B = \bigoplus B_i$ , where each  $B_i$  is the direct sum of uniserial modules of length *i* and *M/B* is *h*-divisible.

For a *QTAG*-module *M*, there is a chain of submodules  $M^0 \supset M^1 \supset M^2 \cdots \supset M^{\tau} = 0$ , for some ordinal  $\tau$ .  $M^{\sigma+1} = (M^{\sigma})^1$ , where  $M^{\sigma}$  is the  $\sigma$ th-*Ulm* submodule of *M*. A fully invariant submodule  $L \subset M$  is a large submodule of *M*, if L + B = M for every basic submodule *B* of *M*. It was proved that several results which hold for *TAG*-modules also hold good for *QTAG*-modules [2]. Notations and terminology are followed from [3].

The Ulm-sequence of x is defined as  $U(x) = (H(x), H(x_1), H(x_2), ...)$ . This is analogous to the U-sequences in groups [4]. These sequences are partially ordered because  $U(x) \le U(y)$  if  $H(x_i) \le H(y_i)$  for every *i*. Transitive and fully transitive QTAG-modules are defined with the help of U-sequences. Ulm invariants and Ulm sequences play an important role in the study of QTAG-modules. Using these concepts transitive and fully transitive modules were defined in [5]. A QTAG-module M is fully transitive if for x,  $y \in M$ ,  $U(x) \le U(y)$ , there is an endomorphism f of M such that f(x) = f(y) and it is transitive if for any two elements x,  $y \in M$ , with  $U(x) \le U(y)$ , there is an automorphism f of M such that f(x) = f(y).

#### 2. Main results

Mehdi et al. characterized fully invariant submodules and characteristic submodules with the help of their socles and define socle-regular and strongly socle-regular *QTAG*-modules [6,7]. We start by recalling their definitions:

A *QTAG*-module *M* is said to be socle-regular (respectively strongly socle-regular) if for all fully invariant (respectively characteristic) submodules *K* of *M*, there exists an ordinal  $\sigma$  (depending on *K*) such that  $Soc(K) = Soc(H_{\sigma}(M))$ ). It is self evident that strongly socle-regular *QTAG*-modules are themselves socle-regular.

**Definition 2.1.** A submodule N of a *QTAG*-module M is said to be projection-invariant in M if  $f(N) \subseteq N$  for all idempotent endomorphisms f in End(M). Clearly, fully invariant submodules are projection-invariant, but the converse is not true in general [8].

It is easy to show that N is projection-invariant in M if and only if,  $f(N) = N \cap f(M)$  for every projection  $f \in End(M)$ . Projection-invariant submodules satisfies the property of being distributed across the direct sum *i.e.*, if  $M = P \oplus Q$  and N is projection-invariant, then  $N = (P \cap N) \oplus (Q \cap N)$  [8].

Motivated by the concepts of socle-regular and strongly socle-regular *QTAG*-modules we make the following definition:

**Definition 2.2.** A *QTAG*-module *M* is said to be projectively socle-regular if for each projection-invariant submodule *N* of *M*, there is an ordinal  $\sigma$  (depending on *N*) such that  $\text{Soc}(N) = \text{Soc}(H_{\sigma}(M))$ .

It is obvious that projectively socle-regular *QTAG*-modules are socle-regular.

Let us recall the terminology used in [6]:

For a submodule N of M, put  $\sigma = \min\{H_M(x)|x \in Soc(N)\}\)$  and denote  $\sigma = \inf(Soc(N))$ . Here  $Soc(N) \subseteq Soc(H_{\sigma}(M))$ .

**Proposition 2.1.** If N is a projection-invariant submodule of a QTAG-module M and  $\inf(\operatorname{Soc}(N)) = k$ , a positive integer, then  $\operatorname{Soc}(N) = \operatorname{Soc}(H_k(M))$ . Consequently, if M is separable, then M is projectively socle-regular.

**Proof.** Suppose that N is a projection-invariant submodule of M and  $\inf(\operatorname{Soc}(N)) = k < \omega$ . It remains to show that  $Soc(H_k(M)) \subseteq Soc(N)$ . As inf(Soc(N)) = k, there is an element  $x \in \text{Soc}(N)$  such that  $H_M(x) = k$  and so  $d(\frac{yR}{xR}) = k$ , for  $y \in M$ . Since every element of exponent one and finite height can be embedded in a direct summand, by [9] yR is a summand of M containing x. Therefore  $M = yR \oplus M'$ , for some submodule M' of M. If z is an arbitrary element of  $Soc(H_k(M))/Soc(H_{k+1}(M))$ , then there exists  $u \in H^{k+1}(M)$ such that  $d(\frac{uR}{R}) = k$  and hence  $M = uR \oplus M''$ . Now, d(uR) =d(yR) = k + 1, implying that  $uR \cong yR$ . Then we have that u =ry + m', for some  $r \in R$  and  $m' \in M'$ . We may define  $\phi : yR \oplus$  $M' \to yR \oplus M'$  by  $\phi(y) = m'$ ,  $\phi(M') = 0$ . Now,  $\phi$  is the difference of two idempotent endomorphisms of M and we define  $\theta: M \to M$  by  $\theta(m) = r(\psi(m)) + \phi(m)$ , where  $\psi$  is the projection map given by  $\psi(y) = y$ ,  $\psi(M') = 0$ . Here  $\theta$  is a sum of idempotents and  $\theta(y) = ry + m' = u$ . Since  $\theta(x) = v$  such that  $d(\frac{vR}{\theta(yR)}) = k$  and vR = zR as  $d(\frac{uR}{zR}) = k$  and  $x \in N$ , which is a projection-invariant submodule of M, we conclude that  $z \in$  $\operatorname{Soc}(N)$ . Hence  $\operatorname{Soc}(H_k(M))/\operatorname{Soc}(H_{k+1}(M)) \subseteq \operatorname{Soc}(N)$ . However, if  $s \in \text{Soc}(H_{k+1}(M))$ , then  $z + s \in \text{Soc}(H_k(M))$  and so by the argument above,  $z + s \in Soc(N)$ . Thus we have that  $Soc(H_k(M)) \subseteq Soc(N)$  and we are done.  $\Box$ 

**Corollary 2.1.** If M is a QTAG-module such that  $d(H_{\omega}(M)) = 1$ , then M is projectively socle-regular.

**Proof.** Suppose N is a projection-invariant submodule of M. If  $\operatorname{Soc}(N) \nsubseteq H_{\omega}(M)$ , then  $\inf(\operatorname{Soc}(N))$  is finite and by Proposition 2.1 above we obtain that  $\operatorname{Soc}(N) = \operatorname{Soc}(H_k(M))$ for some integer k. So we may assume that  $\operatorname{Soc}(N) \subseteq H_{\omega}(M)$ . Since the  $H_{\omega}(M)$  is a uniserial module of decomposition length 1, either  $N + \operatorname{Soc}(N) = 0$  whence  $\operatorname{Soc}(N) = \operatorname{Soc}(H_{\omega+1}(M))$  or  $\operatorname{Soc}(N) = \operatorname{Soc}(H_{\omega}(M))$  as required.  $\Box$ 

The property of a *QTAG*-module *M* being projectively socleregular is inherited by submodules of the form  $H_{\sigma}(M)$ .

**Proposition 2.2.** If M is a projectively socle-regular QTAGmodule, then so also is  $H_{\sigma}(M)$ , for all ordinals  $\sigma$ .

**Proof.** Let  $K = H_{\sigma}(M)$  and suppose that N is a projection invariant submodule of K. Let f be an arbitrary idempotent in End(M). Then  $f^* = f|_K$  is an idempotent endomorphism of K. Thus  $f(N) = f^*(N) \subseteq N$ , since N is projectioninvariant submodule of K. Consequently N is a projectioninvariant submodule of M and so there is an ordinal  $\rho$  such that Download English Version:

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