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Original Article

On projection-invariant submodules of QTAG-modules

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Abstract A module M over an associative ring R with unity is a $QTAG$-module if every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules. Here we study projection-invariant submodule of $QTAG$-module. A submodule N of a $QTAG$-module M is said to be projection-invariant in M if $f(N) \subseteq N$, for all idempotent endomorphisms f in End(M). Clearly, fully invariant submodules are projection-invariant. Mehdi et. al. characterized fully invariant submodules and characteristic submodules with the help of their socles. Here we investigate the socles of projection-invariant submodules of $QTAG$-modules.

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1. Introduction and preliminaries

All the rings R considered here are associative with unity and modules M are unital $QTAG$-modules. An element $x \in M$ is uniform, if xR is a non-zero uniform (hence uniserial) module and for any R-module M with a unique composition series, $d(M)$ denotes its composition length. For a uniform element $x \in M, e(x) = d(xR)$ and $H_M(x) = \sup\{d(\frac{yR}{xR}) | y \in M,\ x \in yR$ and y uniform$\}$ are the exponent and height of x in M, respectively. $H_k(M)$ denotes the submodule of M generated by the elements of height at least k and $H^k(M)$ is the submodule of M generated by the elements of exponents at most k. M is h-divisible if $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$ and it is h-reduced if it does not contain any h-divisible submodule. In other words it is free from the elements of infinite height. A $QTAG$-module M is said to be separable, if $M^1 = 0$. A family $\mathcal{N}$ of submodules of M is called a nice system in M if.

(i) $0 \in \mathcal{N}$;
(ii) If $\{N_i\}_{i \in I}$ is any subset of $\mathcal{N}$, then $\Sigma_I N_i \in \mathcal{N}$;
(iii) Given any $N \in \mathcal{N}$ and any countable subset X of M, there exists $K \in \mathcal{N}$ containing $N \cup X$, such that K/N is countably generated [1].

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A h-reduced $QTAG$-module M is called totally projective if it has a nice system. A submodule $B \subseteq M$ is a basic submodule of M, if B is h-pure in M, $B = \oplus B_i$, where each B_i is the direct sum of uniserial modules of length i and M/B is h-divisible.

For a $QTAG$-module M, there is a chain of submodules $M^0 \supset M^1 \supset M^2 \cdots \supset M^\tau = 0$, for some ordinal τ. $M^{\sigma+1} = (M^\sigma)^1$, where M^σ is the σth-*Ulm* submodule of M. A fully invariant submodule $L \subset M$ is a large submodule of M, if $L + B = M$ for every basic submodule B of M. It was proved that several results which hold for TAG-modules also hold good for $QTAG$-modules [2]. Notations and terminology are followed from [3].

The *Ulm*-sequence of x is defined as $U(x) = (H(x), H(x_1), H(x_2), \ldots)$. This is analogous to the U-sequences in groups [4]. These sequences are partially ordered because $U(x) \le U(y)$ if $H(x_i) \le H(y_i)$ for every i. Transitive and fully transitive $QTAG$-modules are defined with the help of U-sequences. *Ulm* invariants and *Ulm* sequences play an important role in the study of $QTAG$-modules. Using these concepts transitive and fully transitive modules were defined in [5]. A $QTAG$-module M is fully transitive if for $x, y \in M$, $U(x) \le U(y)$, there is an endomorphism f of M such that $f(x) = f(y)$ and it is transitive if for any two elements $x, y \in M$, with $U(x) \le U(y)$, there is an automorphism f of M such that $f(x) = f(y)$.

2. Main results

Mehdi et al. characterized fully invariant submodules and characteristic submodules with the help of their socles and define socle-regular and strongly socle-regular $QTAG$-modules [6,7]. We start by recalling their definitions:

A $QTAG$-module M is said to be socle-regular (respectively strongly socle-regular) if for all fully invariant (respectively characteristic) submodules K of M, there exists an ordinal σ (depending on K) such that $\mathrm{Soc}(K) = \mathrm{Soc}(H_\sigma(M))$). It is self evident that strongly socle-regular $QTAG$-modules are themselves socle-regular.

Definition 2.1. A submodule N of a $QTAG$-module M is said to be projection-invariant in M if $f(N) \subseteq N$ for all idempotent endomorphisms f in $\mathrm{End}(M)$. Clearly, fully invariant submodules are projection-invariant, but the converse is not true in general [8].

It is easy to show that N is projection-invariant in M if and only if, $f(N) = N \cap f(M)$ for every projection $f \in \mathrm{End}(M)$. Projection-invariant submodules satisfies the property of being distributed across the direct sum *i.e.*, if $M = P \oplus Q$ and N is projection-invariant, then $N = (P \cap N) \oplus (Q \cap N)$ [8].

Motivated by the concepts of socle-regular and strongly socle-regular $QTAG$-modules we make the following definition:

Definition 2.2. A $QTAG$-module M is said to be projectively socle-regular if for each projection-invariant submodule N of M, there is an ordinal σ (depending on N) such that $\mathrm{Soc}(N) = \mathrm{Soc}(H_\sigma(M))$.

It is obvious that projectively socle-regular $QTAG$-modules are socle-regular.

Let us recall the terminology used in [6]:

For a submodule N of M, put $\sigma = \min\{H_M(x) | x \in \mathrm{Soc}\ (N)\}$ and denote $\sigma = \inf(\mathrm{Soc}(N))$. Here $\mathrm{Soc}(N) \subseteq \mathrm{Soc}(H_\sigma(M))$.

Proposition 2.1. *If N is a projection-invariant submodule of a $QTAG$-module M and $\inf(\mathrm{Soc}(N)) = k$, a positive integer, then $\mathrm{Soc}(N) = \mathrm{Soc}(H_k(M))$. Consequently, if M is separable, then M is projectively socle-regular.*

Proof. Suppose that N is a projection-invariant submodule of M and $\inf(\mathrm{Soc}(N)) = k < \omega$. It remains to show that $\mathrm{Soc}(H_k(M)) \subseteq \mathrm{Soc}(N)$. As $\inf(\mathrm{Soc}(N)) = k$, there is an element $x \in \mathrm{Soc}(N)$ such that $H_M(x) = k$ and so $d(\frac{yR}{xR}) = k$, for $y \in M$. Since every element of exponent one and finite height can be embedded in a direct summand, by [9] yR is a summand of M containing x. Therefore $M = yR \oplus M'$, for some submodule M' of M. If z is an arbitrary element of $\mathrm{Soc}(H_k(M))/\mathrm{Soc}(H_{k+1}(M))$, then there exists $u \in H^{k+1}(M)$ such that $d(\frac{uR}{zR}) = k$ and hence $M = uR \oplus M''$. Now, $d(uR) = d(yR) = k + 1$, implying that $uR \cong yR$. Then we have that $u = ry + m'$, for some $r \in R$ and $m' \in M'$. We may define $\phi : yR \oplus M' \to yR \oplus M'$ by $\phi(y) = m'$, $\phi(M') = 0$. Now, ϕ is the difference of two idempotent endomorphisms of M and we define $\theta : M \to M$ by $\theta(m) = r(\psi(m)) + \phi(m)$, where ψ is the projection map given by $\psi(y) = y$, $\psi(M') = 0$. Here θ is a sum of idempotents and $\theta(y) = ry + m' = u$. Since $\theta(x) = v$ such that $d(\frac{vR}{\theta(yR)}) = k$ and $vR = zR$ as $d(\frac{uR}{zR}) = k$ and $x \in N$, which is a projection-invariant submodule of M, we conclude that $z \in \mathrm{Soc}(N)$. Hence $\mathrm{Soc}(H_k(M))/\mathrm{Soc}(H_{k+1}(M)) \subseteq \mathrm{Soc}(N)$. However, if $s \in \mathrm{Soc}(H_{k+1}(M))$, then $z + s \in \mathrm{Soc}(H_k(M))$ and so by the argument above, $z + s \in \mathrm{Soc}(N)$. Thus we have that $\mathrm{Soc}(H_k(M)) \subseteq \mathrm{Soc}(N)$ and we are done. □

Corollary 2.1. *If M is a $QTAG$-module such that $d(H_\omega(M)) = 1$, then M is projectively socle-regular.*

Proof. Suppose N is a projection-invariant submodule of M. If $\mathrm{Soc}(N) \nsubseteq H_\omega(M)$, then $\inf(\mathrm{Soc}(N))$ is finite and by Proposition 2.1 above we obtain that $\mathrm{Soc}(N) = \mathrm{Soc}(H_k(M))$ for some integer k. So we may assume that $\mathrm{Soc}(N) \subseteq H_\omega(M)$. Since the $H_\omega(M)$ is a uniserial module of decomposition length 1, either $N + \mathrm{Soc}(N) = 0$ whence $\mathrm{Soc}(N) = \mathrm{Soc}(H_{\omega+1}(M))$ or $\mathrm{Soc}(N) = \mathrm{Soc}(H_\omega(M))$ as required. □

The property of a $QTAG$-module M being projectively socle-regular is inherited by submodules of the form $H_\sigma(M)$.

Proposition 2.2. *If M is a projectively socle-regular $QTAG$-module, then so also is $H_\sigma(M)$, for all ordinals σ.*

Proof. Let $K = H_\sigma(M)$ and suppose that N is a projection invariant submodule of K. Let f be an arbitrary idempotent in $\mathrm{End}(M)$. Then $f^* = f|_K$ is an idempotent endomorphism of K. Thus $f(N) = f^*(N) \subseteq N$, since N is projection-invariant submodule of K. Consequently N is a projection-invariant submodule of M and so there is an ordinal ρ such that