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Original Article

Congruences and *d*-filters of principal *p*-algebras

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Abstract The concept of *d*-filters is introduced in *p*-algebras. Some properties of *d*-filters are studied. It is proved that the class $F^d(L)$ of all *d*-filters of a *p*-algebra L is a bounded complete lattice. A characterization of *d*-filters of a principal *p*-algebra is given. Also many properties of congruences induced by the *d*-filters are derived. A relationship between the *d*-filters of a principal *p*-algebra L and the congruences in $[\Phi, \nabla]$ is established.

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1. Introduction

The notion of pseudo-complements was introduced in semilattices and distributive lattices by O. Frink [1] and G. Birkhoff [2]. The pseudo-complements in Stone algebras were studied and discussed by O. Frink [1], R. Balbes [3] and G. Gratzer [4] etc. Recently, the concept of Boolean filter of bounded pseudo-complemented distributive lattices was introduced by M. Sambasiva Rao and K. P. Shum in [5]. A. Badawy and K. P. Shum [6] introduced and characterized the congruences and Boolean filters of quasi-modular *p*-algebras. Also A. Badawy and M. Sambasiva Rao [7] studied σ-ideals of distributive *p*-algebras. A. Badawy and M. Atallah [8] introduced the notion of Boolean filters of principal *p*-algebras

In this paper, we further study the *d*-filters in a *p*-algebra L and many properties of *d*-filters are also given. We will give a characterization theorem of *d*-filters of a principal *p*-algebra L. We also notice that the set $F^d(L)$ of all *d*-filters of a *p*-algebra forms a complete lattice. The relationship between the *d*-filters and the congruences in $[\Phi, \nabla]$ of a principal *p*-algebra L is introduced. We also prove that the Boolean algebras $B(L)$ and $Con_B(L) = \{\theta_a : a \in B(L)\}$ are isomorphic, where θ_a is the congruence on L induced by a *d*-filter $[a)^d$ for a closed element a of L. Moreover, we show that the Boolean algebra $Con_B(L)$ can be embedded into the interval $[\Phi, \nabla]$ of $Con(L)$. It is proved that the lattice of all *d*-filters of a finite principal *p*-algebra L is isomorphic to the sublattice $[\Phi, \nabla]$ of $Con(L)$.

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2. Preliminaries

In this section, we cite some known definitions and basic results which can be found in the papers [1,9–13].

A p-algebra is a universal algebra $(L, \vee, \wedge, ^*, 0, 1)$, where $(L, \vee, \wedge, 0, 1)$ is a bounded lattice and the unary operation * is defined by $x \wedge a = 0 \Leftrightarrow x \leq a^*$.

It is known that the class of all p-algebras is equational. A quasi-modular p-algebra is a p-algebra satisfying the identity

$$((x \wedge y) \vee z^{**}) \wedge x = (x \wedge y) \vee (z^{**} \wedge x).$$

An element a of a p-algebra L is called closed if $a^{**} = a$. Then $B(L) = \{a \in L : a = a^{**}\}$ is the set of all closed elements of L. It is known that $(B(L), \triangledown, \wedge, 0, 1)$, where $a \triangledown b = (a^* \wedge b^*)^*$, forms a Boolean algebra. The set $D(L) = \{x \in L : x^* = 0\} = \{x \vee x^* : x \in L\}$ of all dense elements of L is a filter of L.

For an arbitrary lattice L, the set $F(L)$ of all filters of L ordered by the set inclusion forms a lattice. It is known that $F(L)$ is modular (distributive) if and only if L is a modular (distributive) lattice. Let $a \in L$ and $[a)$ be the principal filter of L generated by $a : [a) = \{x \in L : x \geq a\}$.

An equivalent relation θ on a p-algebra $(L; \vee, \wedge, ^*)$ is called a congruence relation if

(1) θ is a lattice congruence, i.e., for all $(x, y), (x_1, y_1) \in \theta$ implies $(x \wedge x_1, y \wedge y_1), (x \vee x_1, y \vee y_1) \in \theta$,
(2) $(x, y) \in \theta$ implies $(x^*, y^*) \in \theta$.

Through what follows, for a p-algebra L we shall denote by ∇ the universal congruence on L. The Cokernel of the lattice congruence θ on a lattice L is defined as

$$Coker\theta = \{x \in L : (x, 1) \in \theta\}.$$

The relation Φ of a p-algebra L is defined by $(x, y) \in \Phi \Leftrightarrow x^{**} = y^{**}$ and is called the Glivenko congruence relation. It is known that the Glivenko congruence is indeed a congruence on L such that $L/\Phi \cong B(L)$ holds.

We frequently use the following rules in the computations of p-algebras (see [10,13]):

(1) $0^{**} = 0$ and $1^{**} = 1$,
(2) $a \wedge a^* = 0$;
(3) $a \leq b$ implies $b^* \leq a^*$,
(4) $a \leq a^{**}$,
(5) $a^{***} = a^*$,
(6) $(a \vee b)^* = a^* \wedge b^*$,
(7) $(a \wedge b)^* \geq a^* \vee b^*$,
(8) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$,
(9) $(a \vee b)^{**} = (a^* \wedge b^*)^* = (a^{**} \vee b^{**})^{**}$.

Haviar [14] introduced the class of principal p-algebras which contains all quasi-modular p-algebras having a smallest dense element.

Definition 2.1 ([14]). A p-algebra $(L; \vee, \wedge, ^*, 0, 1)$ is called a principal p-algebra, if it satisfies the following conditions:

(i) The filter $D(L)$ is principal, i.e., there exists an element $d \in L$ such that $D(L) = [d)$,
(ii) The element d is distributive, i.e., $(x \wedge y) \vee d = (x \vee d) \wedge (y \vee d)$ for all $x, y \in L$,
(iii) $x = x^{**} \wedge (x \vee d)$ for any $x \in L$.

Throughout this paper, d stands for a smallest dense element of a principal p-algebra L, unless otherwise mentioned.

3. Properties of d-filters

In this section, we introduce the concept of d-filter of a p-algebra. Some properties of d-filters in a p-algebra are derived. A characterization theorem of d-filters of a principal p-algebra will be given.

Definition 3.1. For any filter F of a p-algebra L, define an extension of F as the set

$$F^d = \{x \in L : x^{**} \geq f \text{ for some } f \in F\}$$

The following two Lemmas represent some basic properties of the set F^d.

Lemma 3.2. *The set F^d is a filter of a p-algebra L containing F.*

Proof. Clearly $1 \in F^d$. Let $x, y \in F^d$. Then $x^{**} \geq f$ and $y^{**} \geq g$ for some f, g of F. Hence $(x \wedge y)^{**} = x^{**} \wedge y^{**} \geq f \wedge g$. It follows that $x \wedge y \in F^d$ as $f \wedge g \in F$. Now, let $z \in L$ be such that $z \geq x \in F^d$. Then $z^{**} \geq x^{**} \geq f$ for some $f \in F$. Hence $z \in F^d$. Therefore F^d is a filter of L. Since $x^{**} \geq x$ for any $x \in F$, we have that $x \in F^d$ and $F \subseteq F^d$. □

Lemma 3.3. *For any two filters F, G of a p-algebra L, we have the following:*

(1) *$F \subseteq G$ implies $F^d \subseteq G^d$,*
(2) *$(F \cap G)^d = F^d \cap G^d$,*
(3) *$(F^d)^d = F^d$.*

Proof.

(1) Suppose that $F \subseteq G$. Let $x \in F^d$. Then, $x^{**} \geq f$ for some $f \in F$. It follows that $x \in G^d$ as $f \in G$.
(2) Obviously $(F \cap G)^d \subseteq F^d \cap G^d$. Conversely, let $x \in F^d \cap G^d$. Then $x^{**} \geq f$ and $x^{**} \geq g$ for some $f, g \in F$. Hence $x^{**} \geq f \vee g$. It yields that $x \in (F \cap G)^d$, where $f \vee g \in F \cap G$. Consequently $F^d \cap G^d \subseteq (F \cap G)^d$.
(3) By (1) above, $F^d \subseteq (F^d)^d$. Conversely, let $x \in (F^d)^d$. Then $x^{**} \geq f$ for some $f \in F^d$. Since $f \in F^d$, we have $f^{**} \geq f_1$ for some $f_1 \in F$. Hence $x^{**} \geq f^{**} \geq f_1$. Then $x \in F^d$ as $f_1 \in F$.

□

We now introduce the concept of d-filters in a p-algebra.

Definition 3.4. A filter F of a p-algebra L is called an d-filter of L if it satisfies the condition, $F = F^d$.

From Lemma 3.3(2), we can observe that the intersection of two d-filters of a p-algebra is again a d-filter. But, in general, the supremum of two d-filters need not be a d-filter. However, in the following, we obtain the class $F^d(L)$ of all d-filters of L that is a bounded lattice.

Theorem 3.5. *For any p-algebra L, the class $F^d(L)$ forms a complete lattice on its own.*

Proof. For any two d-filters F, G of L, define the ordering $\leq$ on $F^d(L)$ such that $F \leq G \Leftrightarrow F \subseteq G$. Then clearly $(F^d(L), \leq)$ is a partially ordered set. Now, consider the following:

$$F \cap G = (F \cap G)^d \text{ and } F \sqcup G = (F \vee G)^d.$$

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