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Original Article

A note on the qualitative behaviors of non-linear Volterra integro-differential equation

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Abstract This paper considers a scalar non-linear Volterra integro-differential equation. We establish sufficient conditions which guarantee that the solutions of the equation are stable, globally asymptotically stable, uniformly continuous on $[0, \infty)$, and belongs to $L^1[0, \infty)$ and $L^2[0, \infty)$ and have bounded derivatives. We use the Lyapunov's direct method to prove the main results. Examples are also given to illustrate the importance of our results. The results of this paper are new and complement previously known results.

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1. Introduction

In 2009, Becker [1] considered the scalar linear homogeneous Volterra integro-differential equation

$$x'(t) = -a(t)x(t) + \int_0^t b(t, s)x(s)ds, \tag{1}$$

for $t \geq 0$, where a and b are real-valued and continuous functions on the respective domains $[0, \infty)$ and $\Omega := \{(t, s) : 0 \leq s \leq t < \infty\}$. The author studied the asymptotic behaviors of solutions of Eq. (1) by using the Lyapunov functionals.

In this paper, instead of Eq. (1), we consider the non-linear Volterra integro-differential equation of the form

$$x'(t) = -a(t)h(x(t)) + \int_0^t b(t, s)g(x(s))ds, \tag{2}$$

for $t \geq 0$, where $a : [0, \infty) \to [0, \infty)$, $h : \mathfrak{R} \to \mathfrak{R}, g : \mathfrak{R} \to \mathfrak{R}$ and $b : \Omega \to \mathfrak{R}$ are continuous functions on their respective domains, $\Omega := \{(t, s) : 0 \leq s \leq t < \infty\}$, that $h(0) = g(0) = 0$, and $h(x)$ and $g(x)$ are differentiable at $x = 0$.

We can write Eq. (2) in the form of

$$x'(t) = -a(t)h_1(x(t))x(t) + \int_0^t b(t, s)g_1(x(s))x(s)ds,$$

where

$$h_1(x) = \begin{cases} \dfrac{h(x)}{x}, & x \neq 0 \\ h'(0), & x = 0 \end{cases}$$

and

$$g_1(x) = \begin{cases} \dfrac{g(x)}{x}, & x \neq 0 \\ g'(0), & x = 0. \end{cases}$$

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In view of the mentioned information, it follows that the equation discussed by Becker [1], Eq. (1), is a special case of our equation, Eq. (2). That is, our equation, Eq. (2), includes Eq. (1) discussed in [1]. As Becker [1] studied the asymptotic behavior of solutions of linear Volterra integro-differential equation, we investigate the same topic for the nonlinear case. This case shows the novelty of this paper and is an improvement on the topic in the literature. Our results will also be different from that obtained in the literature (see [2–11,13–19] and the references thereof). Namely, the equation considered and the assumptions to be established here are different from that in the mentioned papers above. It should be noted that this paper has also a contribution to the subject in the literature, and it may be useful for researchers working on the qualitative behaviors of solutions for Volterra integro-differential equation.

We give some basic information related Eq. (2) and the non-homogeneous equation

$$x'(t) = -a(t)h(x(t)) + \int_0^t b(t,s)g(x(s))ds + f(t), \tag{3}$$

where $f : [0,\infty) \to \mathfrak{R}$ is a continuous function. It is worth mentioning that the following basic notations and definitions were taken from Becker [1].

We use the following notation throughout this paper (see [1]):

$C[t_0, t_1]$ (resp. $C[t_0, \infty)$) will denote the set of all continuous real-valued functions on $[t_0, t_1]$ (resp. $[t_0, \infty)$).

For $\varphi \in C[0, t_0]$, $|\varphi|_{t_0} := \sup\{|\varphi(t)| : 0 \le t \le t_0\}$.

$L^1[0,\infty)$ denotes the set of all real-valued functions f that are Lebesgue measurable on $[0,\infty)$ and for which the Lebesgue integral $\int_0^\infty |f|$ is finite. However, we use it to denote those functions in $L^1[0,\infty)$ that are also continuous on $[0,\infty)$. For such a function, say g, the improper Riemann integral $\int_0^\infty |g(t)|dt$ converges, i.e., $\lim_{t\to\infty} \int_0^t |g(s)|ds$ exists and is finite. In short, by $g \in L^1[0,\infty)$ we mean that g is continuous and absolutely Riemann integrable on $[0,\infty)$.

$L^2[0,\infty)$ will denote the set of all continuous real-valued functions that are square integrable on $[0,\infty)$. That is, $h \in L^2[0,\infty)$ will mean that h is continuous on $[0,\infty)$ and $h^2 \in L^1[0,\infty)$.

Definition 1. A solution of Eq. (2) (resp. (3)) on $[0, T)$, where $0 < T \le \infty$, with an initial value $x_0 \in \mathfrak{R}$ is a continuous function $x : [0, T) \to \mathfrak{R}$ that satisfies Eq. (2) (resp. (3)) on $[0, T)$ such that $x(0) = x_0$.

Definition 2. The zero solution of Eq. (2) is

(i) stable if for each $\varepsilon > 0$ and $t_0 \ge 0$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that $\varphi \in C[0, t_0]$ with $|\varphi|_{t_0} < \delta$ implies that $|x(t, t_0, \varphi)| < \varepsilon$ for all $t \ge t_0$.

(ii) globally asymptotically stable (asymptotically stable in the large) if it is stable and if every solution of Eq. (2) approaches zero as $t \to \infty$.

(iii) uniformly stable if for each $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $\varphi \in C[0, t_0]$ with $|\varphi|_{t_0} < \delta$ (any $t_0 \ge 0$) implies that $|x(t, t_0, \varphi)| < \varepsilon$ for all $t \ge t_0$.

(iv) uniformly asymptotically stable if it is uniformly stable and if there exits an $\eta > 0$ with the following property: For each $\varepsilon > 0$, there exists a $T = T(\varepsilon) > 0$ such that $\varphi \in C[0, t_0]$ with $|\varphi|_{t_0} < \eta$ (any $t_0 \ge 0$) implies that $|x(t, t_0, \varphi)| < \varepsilon$ for all $t \ge t_0 + T$.

2. Main results

At the beginning, we obtain some sufficient conditions so that all of the solutions of Eq. (2) belong to $L^2[0,\infty)$. Then we will add more conditions that will drive these L^2 solutions tend to zero as $t \to \infty$.

Lemma 1. *If*

$$1 \le h_1(x) \le \alpha, \quad \sigma \le g_1(x) \le 1,$$

where α and σ, $\sigma \in (0,1)$, are positive constants,

$$a(t) - \int_0^t |b(t,s)|ds \ge 0$$

for all $t \ge 0$ and if

$$\alpha a(s) - \sigma \int_s^t |b(u,s)|du \ge 0$$

for all $t \ge s \ge 0$, then the zero solution of Eq. (2) is stable. In addition, if for some $t_1 \ge 0$ there is a constant $k > 0$ such that either

$$1 \le h_1(x) \le \alpha, \quad \sigma \le g_1(x) \le 1,$$

$$a(t) - \int_0^t |b(t,s)|ds \ge k$$

for all $t \ge t_1$ or

$$1 \le h_1(x) \le \alpha, \quad \sigma \le g_1(x) \le 1,$$

$$\alpha a(s) - \sigma \int_s^t |b(u,s)|du \ge k$$

for all $t \ge s \ge t_1$ holds, then every solution $x(t)$ of Eq. (2) belongs to $L^2[0,\infty)$.

Proof. We define the Lyapunov functional

$$V : [0,\infty) \times C[0,\infty) \to [0,\infty)$$

by

$$V(t,\psi(.)) := \psi^2(t) + \int_0^t \{a(s)h_1(\psi(s)) - \int_s^t |b(u,s)g_1(\psi(s))|du\}\psi^2(s)ds. \tag{4}$$

It is clear from (4) that $V(t, 0) = 0$ *and* $V(t,\psi(.)) \ge \psi^2(t)$ for all $t \ge 0$.

For any $t_0 \ge 0$ and initial function $\varphi \in C[0, t_0]$, *let* $x(t) = x(t, t_0, \varphi)$ denote the unique solution of Eq. (2) on $[0,\infty)$ such that $x(t) = \varphi(t)$ *for* $0 \le t \le t_0$. For brevity, let $V(t) = V(t, x(.))$, that is, the value of the functional V along the solution $x(t)$ *at* t. Taking the derivative of V with respect to t, we have

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