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Original Article

Simple equation method for nonlinear partial differential equations and its applications

Taher A. Nofal*

Department of Mathematics and Statistics, Faculty of Science, Taif University, Taif, Saudi Arabia
Mathematics Department, Faculty of Science, Minia University, Minia, Egypt

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Abstract In this article, we focus on the exact solution of the some nonlinear partial differential equations (NLPDEs) such as, Kodomtsev–Petviashvili (KP) equation, the (2 + 1)-dimensional breaking soliton equation and the modified generalized Vakhnenko equation by using the simple equation method. In the simple equation method the trial condition is the Bernoulli equation or the Riccati equation. It has been shown that the method provides a powerful mathematical tool for solving nonlinear wave equations in mathematical physics and engineering problems.

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1. Introduction

The nonlinear partial differential equations (NLPDEs) play an important role to study many problems in physics and geometry. The effort in finding exact solutions to nonlinear equations is important for the understanding of most nonlinear physical phenomena [1,2]. Nonlinear wave phenomena appears in various scientific and engineering fields, such as fluid mechanics, plasma physics, optimal fiber, biology, oceanology, solid state physics, chemical physics and geometry. In recent years, the powerful and efficient methods to find analytic solutions of nonlinear equations have drawn a lot of interest by a diverse group of scientists, such as the inverse scattering transform method [3], Backland transformation [4], Darboux transformation [5], Hirota bilinear method [6], variable separation approach [7], various tanh method [11,8–10], homogenous balance method [12], similarity reductions method [13,14], (G'/G)-expansion method [15,16], sine–cosine method [17], the exp-function method [18], the sub-ODE method [19], and so on.

In this paper, we obtain the exact solution of Kodomtsev–Petviashvili (KP) equation, the $(2+1)$-dimensional breaking soliton equation and the modified generalized Vakhnenko equation by using the simple equation method. The simple equation method is a very powerful mathematical technique for finding exact solution of nonlinear ordinary differential equations. It has been developed by Kadreyshov [20,21] and

* Address: Department of Mathematics and Statistics, Faculty of Science, Taif University, Taif, Saudi Arabia.
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used successfully by many authors for finding exact solution of ODEs in mathematical physics [22,23]. In Section 2, we give a brief algorithm for the simple equation method. In Section 3, we apply this method to KP equation, the (2+1)-dimensional breaking soliton equation and the modified generalized Vakhnenko equation. We give the conclusion in Section 4.

2. Algorithm of the simple equation method

In this section we will describe a direct method namely simple equation method for finding the traveling wave solution of nonlinear evolution equations. Suppose that the nonlinear partial equation, in two independent variables x and t is given by:

$$P(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \dots) = 0, \tag{2.1}$$

where $u(x,t)$ is an unknown function, P is a polynomial of $u(x,t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following we give the main steps of this method.

Step (1): Combining the independent variables x and t into one variable $\xi = x - ct$, we suppose that

$$u(x,t) = u(\xi), \quad \xi = x - ct. \tag{2.2}$$

The traveling wave transformation Eq. (2.2) permits us to reduce Eq. (2.1) to the following ordinary differential equation (ODE)

$$Q(u, u', u'', \dots) = 0 \tag{2.3}$$

where Q is a polynomial of $u(\xi)$ and its derivatives, where $u'(\xi) = \frac{du}{d\xi}, u''(\xi) = \frac{d^2u}{d\xi^2}$ and so on.

Step (2): We seek the solution of Eq. (2.2) in the following form:

$$u(\xi) = \sum_{i=0}^{n} a_i F^i(\xi). \tag{2.4}$$

which $a_i (i = 0, 1, 2, \dots, n)$ are constants to be determined later and $F(\xi)$ are the functions that satisfy the simple equations (ordinary differential equations). The simple equation has two properties, first it is lesser order than Eq. (2.2), second we know the general solution of the simple equation. In this paper, we shall use as simple equations, the Bernoulli and Riccati equations which are well known nonlinear ordinary differential equations and their solutions can be expressed by elementary functions. For the Bernoulli equation

$$F'(\xi) = cF(\xi) + dF^2(\xi). \tag{2.5}$$

Step (3): The balance number n can be determined by balancing the highest order derivative and nonlinear terms in Eq. (2.2).

Step (4): We discuss the general solutions of the simple Eq. (2.5) as following:

$$F(\xi) = \frac{c\,\exp[c(\xi + \xi_0)]}{1 - d\,\exp[c(\xi + \xi_0)]}. \tag{2.6}$$

For case $d < 0, c > 0$, here ξ_0 is a constant of the integration, and

$$F(\xi) = -\frac{c\,\exp[c(\xi + \xi_0)]}{1 + d\,\exp[c(\xi + \xi_0)]}. \tag{2.7}$$

For the Riccati equation

$$F'(\xi) = \alpha F^2(\xi) + \beta. \tag{2.8}$$

Eq. (2.8) admits the following exact solutions [23],

$$F(\xi) = -\frac{\sqrt{-\alpha\beta}}{\alpha}\tanh\left(\sqrt{-\alpha\beta}\,\xi - \frac{\nu \ln(\xi_0)}{2}\right), \quad \xi_0 > 0, \quad \nu = \pm 1, \tag{2.9}$$

where $\alpha\beta < 0$ and

$$F(\xi) = \frac{\sqrt{\alpha\beta}}{\alpha}\tan\left(\sqrt{\alpha\beta}(\xi + \xi_0)\right), \quad \xi_0 \text{ is a constant}, \tag{2.10}$$

where $\alpha\beta > 0$.

Remark 1. (i) When $c = \delta, d = -1$ the Eq. (2.5) has another form of Bernoulli equation

$$F'(\xi) = \delta F(\xi) - F^2(\xi), \tag{2.11}$$

which has the exact solutions when $\delta > 0$,

$$F(\xi) = \frac{\delta}{2}\left[1 + \tanh\left(\frac{\delta}{2}(\xi + \xi_0)\right)\right], \tag{2.12}$$

and when $\delta < 0$,

$$F(\xi) = \frac{\delta}{2}\left[1 - \tanh\left(\frac{\delta}{2}(\xi + \xi_0)\right)\right], \tag{2.13}$$

(ii) When $c = 1, d = -1$ the Eq. (2.5) has another form of Riccati equation [20,21]

$$F'(\xi) - F(\xi) + F^2(\xi) = 0, \tag{2.14}$$

which has the logistic function as the exact solutions

$$F(\xi) = \frac{1}{1 + e^{-\delta}}. \tag{2.15}$$

The logistic Eq. (2.15) can be presented in the hyperbolic tangent function according to the relation

$$\frac{1}{1 + e^{-\xi}} = \frac{1}{2}\left[1 + \tanh\left(\frac{\xi}{2}\right)\right]. \tag{2.16}$$

3. Application

3.1. Kodomtsev–Petviashvili (KP) equation

The Kodomtsev–Petviashvili (KP) equation

$$u_{xt} - 6u\,u_{xx} - 6(u_x)^2 + u_{xxxx} + 3\delta^2 u_{yy} = 0 \tag{3.1}$$

or

$$(u_t - 6u\,u_x + u_{xxx})_x + 3\delta^2 u_{yy} = 0$$

is a two-dimensional generalization of the Kdv equation. Kodomtsev and Petviashvili (1970) first introduced this equation to describe slowly varying nonlinear waves in a dispersive