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ORIGINAL ARTICLE

Solutions of some class of nonlinear PDEs in mathematical physics

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Abstract In this work, the modified simple equation (MSE) method is applied to some class of nonlinear PDEs, namely, a system of nonlinear PDEs, a (2 + 1)-dimensional nonlinear model generated by the Jaulent–Miodek hierarchy, and a generalized KdV equation with two power nonlinearities.

As a result, exact traveling wave solutions involving parameters have been obtained for the considered nonlinear equations in a concise manner. When these parameters are chosen as special values, the solitary wave solutions are derived. It is shown that the proposed technique provides a more powerful mathematical tool for constructing exact solutions for a broad variety of nonlinear PDEs in mathematical physics.

Mathematics Subject Classification: 35C05; 35L05; 35Q99

1. Introduction

Recently, several methods have been used to extract exact solutions of nonlinear partial differential equations (NPDEs) such as inverse scattering method [1,2], the Darboux transform [3,4], the Hirota bilinear method [5,6], the Backlund transformation method [7–9], the Exp-function method [10–12], the (G'/G)-expansion method [13–15], the projective Riccati equation method [16], first integral method [17–20], exp $(-\Phi(\xi))$-expansion method [21–23], the functional variable method [24,25], modified simple equation method [26–33] and others. However, up to now, a unified method that can be used to deal with all types of nonlinear PDEs has not been found yet. To obtain more different types of exact solution, the enhancement of these methods is a challenge topic.

This paper is organized as follows: a description of the modified simple equation (MSE) method is presented first [26–33]. This is followed by an application of this method to three distinct model equations, two of them are (2 + 1)-dimensional nonlinear types, namely, a system of nonlinear PDEs and a nonlinear model generated by Jaulent–Miodek hierarchy, and the third is the generalized KdV nonlinear model with two power nonlinearities.

Finally, some conclusions are given.

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2. The MSE method

Consider a general nonlinear PDE in the form of

$$P(u,u_t,u_x,u_{xx},u_{tt},u_{xt},u_{xxx},\dots)=0. \tag{1}$$

where P is a polynomial in its arguments.

Step 1. Seek solitary wave solutions of Eq. (1) by taking

$$u(x,t)=U(z),\quad z=x-ct+\zeta, \tag{2}$$

where ζ is an arbitrary constant, and transform (1) to a nonlinear ordinary differential equation (ODE) as

$$Q(U,U',U'',U''',\dots)=0. \tag{3}$$

where the prime denotes the derivation with respect to z.

Step 2. We suppose that (3) has the formal solution as

$$U(z)=\sum_{i=0}^{M}\alpha_i\left(\frac{\Psi'(z)}{\Psi(z)}\right)^i, \tag{4}$$

where $\alpha_i,(i=0,1,\dots,M)$ are constants to be determined such that $\alpha_M\neq 0$. The function $\Psi(z)$ is an unknown function to be determined later such that $\Psi'(z)\neq 0$.

Step 3. We determine the positive integer M in (4) by balancing the highest order derivatives and the nonlinear terms in (3).

Step 4. We substitute (4) into (3), we calculate all the necessary derivatives $U',U'',U''',\dots$ and then we account the function $\Psi(z)$.
As a result of this substitution, we get a polynomial of $\Psi^{-j}(z)(j=0,1,\dots)$, with the derivatives of $\Psi(z)$. Equating all the coefficients of $\Psi^{-j}(z)(j=0,1,\dots)$, to zero yields a system of equations which can be solved to obtain α_i and $\Psi(z)$. Finally, substituting the values of α_i and $\Psi(z)$ and its derivative $\Psi'(z)$ into (4) leads to exact solutions of (1).

3. Applications

Example 1. A system of nonlinear PDEs given by [34–36] is

$$iu_t+n(u_{xx}+\alpha_1u_{yy})+\beta_1|u|^2u+\gamma_1uv=0 \tag{5}$$

$$\alpha_2v_{tt}+(v_{xx}-\beta_2v_{yy})+\gamma_2\left(|u|^2\right)_{xx}=0 \tag{6}$$

where $n,\alpha_i,\beta_i,\gamma_i(i=1,2)$ are real constants and $n\neq 0,\beta_1\neq 0,\gamma_1\neq 0,\gamma_2\neq 0$. The important cases of (5) and (6) are shown in [36–38] that are the nonlinear Schrodinger equation [36], the Davey–Stewartson (DS) equations [37], and the generalized Zakharov (GZ) equations [38].

At first we are going to extract exact solutions for (5) and (6), for this purpose, using the transformation formula

$$\begin{aligned}&u(x,y,t)=e^{i\theta}\varphi(\xi),v(x,y,t)=\Gamma(\xi),\\&\theta=px+qy+\varepsilon t,\xi=x+cy+dt+\xi_0\end{aligned} \tag{7}$$

where p,q,ε,c,ξ_0 and d are real constants.

Substituting (7) into (5) and (6), we can know that $d=-2n(p+\alpha_1qc)$, and φ,Γ satisfy the equations

$$\begin{aligned}&-(\omega+p^2n+n\alpha_1q^2)\varphi(\xi)+(n+n\alpha_1c^2)\varphi''(\xi)+\beta_1\varphi^3(\xi)\\&\quad+\gamma_1\varphi(\xi)\Gamma(\xi)=0,\end{aligned} \tag{8}$$

$$(\alpha_2d^2-\beta_2c^2+1)\Gamma''(\xi)+\gamma_2(\varphi^2(\xi))''=0. \tag{9}$$

Integrating (9) twice with respect to ξ and considering the constant of integration equal to zero, we find

$$v(x,y,t)=\Gamma(\xi)=-\frac{\gamma_2}{\left(\alpha_2d^2-\beta_2c^2+1\right)}\varphi^2(\xi). \tag{10}$$

Substituting (10) into (8), gives

$$\varphi''(\xi)-\lambda\varphi(\xi)-\mu\varphi^3(\xi)=0, \tag{11}$$

where

$$\lambda=\frac{\omega+p^2n+n\alpha_1q^2}{n+n\alpha_1c^2},\quad \mu=\frac{-\beta_1\left(\alpha_2d^2-\beta_2c^2+1\right)+\gamma_1\gamma_2}{\left(n+n\alpha_1c^2\right)\left(\alpha_2d^2-\beta_2c^2+1\right)}. \tag{12}$$

Balancing $\varphi''(\xi)$ with $\varphi^3(\xi)$ yields $M=1$. Consequently, we have the formal solution as

$$\varphi(\xi)=A_0+A_1\left(\frac{\Psi'(\xi)}{\Psi(\xi)}\right), \tag{13}$$

where A_0 and A_1 are constants to be determined with $A_1\neq 0$. Also the function $\Psi(\xi)$ has to be determined where $\Psi'(\xi)\neq 0$.

It is easy to see that

$$\varphi'(\xi)=A_1\left(\frac{\Psi''}{\Psi}-\frac{\Psi'^2}{\Psi^2}\right), \tag{14}$$

$$\varphi''(\xi)=A_1\left(\frac{\Psi'''}{\Psi}-3\frac{\Psi'\Psi''}{\Psi^2}+2\frac{\Psi'^3}{\Psi^3}\right). \tag{15}$$

Substituting (13)–(15) into (11) and equating all the coefficients of $\Psi^0,\Psi^{-1},\Psi^{-2},\Psi^{-3}$, to zero, we respectively obtain

$$\Psi^0:\ -\lambda A_0-\mu A_0^3=0, \tag{16}$$

$$\Psi^{-1}:\ A_1\Psi'''-\lambda A_1\Psi'-3A_0^2A_1\mu\Psi'=0, \tag{17}$$

$$\Psi^{-2}:\ -3A_1\Psi'\Psi''-3A_0A_1^2\mu\Psi'^2=0, \tag{18}$$

$$\Psi^{-3}:\ 2A_1\Psi'^3-\mu A_1^3\Psi'^3=0, \tag{19}$$

From (16) and (19), we deduce that

$$A_0=0,\quad A_0=\pm\sqrt{\lambda/\mu},\quad A_1=\pm\sqrt{2/\mu} \tag{20}$$

From (17) and (18), we obtain

$$\Psi'''/\Psi''=-\ell_1, \tag{21}$$

where

$$\ell_1=(\lambda+3A_0^2\mu)/(A_0A_1\mu). \tag{22}$$

Integrating (21), we obtain

$$\Psi''=C_1e^{-\ell_1\xi}, \tag{23}$$

where C_1 is a constant of integration.

And from (18) and (23), we obtain

$$\Psi'=-m_1e^{-\ell_1\xi} \tag{24}$$

where

$$m_1=C_1/(A_0A_1\mu). \tag{25}$$

Integrating (24) with respect to ξ, we get

$$\Psi(\zeta)=C_2+(m_1/\ell_1)e^{-\ell_1\xi}, \tag{26}$$

where C_2 is the constant of integration.