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Original Article

$\mathcal{IP}$-separation axioms in ideal bitopological ordered spaces II

A. Kandil[a], O. Tantawy[b], S.A. El-Sheikh[c], M. Hosny[c,*]

[a] *Department of Mathematics, Faculty of Science, Helwan University, Egypt*
[b] *Department of Mathematics, Faculty of Science, Zagazig University, Cairo, Egypt*
[c] *Department of Mathematics, Faculty of Education, Ain Shams University, 11757 Beside Tabary School, Roxy, Cairo, Egypt*

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Abstract The main purpose of this paper was to continue the study of separation axioms which is introduced in part I (Kandil et al., 2015). Whereas the part I (Kandil et al., 2015) was devoted to the axioms $\mathcal{IPT}_i$-ordered spaces, $i = 0, 1, 2$, in the part II the axioms $\mathcal{IPT}_i$-ordered spaces, $i = 3, 4, 5$ and $\mathcal{IPR}_j$-ordered spaces, $j = 2, 3, 4$ are introduced and studied. Clearly, if $\mathcal{I} = \{\phi\}$ in these axioms, then the previous axioms (Singal and Singal, 1971; Abo Elhamayel Abo Elwafa, 2009) coincide with the present axioms. Therefore, the current work is a generalization of the previous one. In addition, the relationships between these axioms and the previous one axioms have been obtained. Some examples are given to illustrate the concepts. Moreover, some important results related to these separations have been obtained.

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1. Introduction

A bitopological space (X, τ_1, τ_2) was introduced by Kelly [4] in 1963, as a method of generalizes topological spaces (X, τ). Every bitopological space (X, τ_1, τ_2) can be regarded as a topological space (X, τ) if $\tau_1 = \tau_2 = \tau$. Furthermore, he extended some of the standard results of separation axioms of topological spaces to bitopological spaces. Thereafter, a large number of papers have been written to generalize topological concepts to bitopological setting.

In 1971 Singal and Singal [2] presented and studied the bitopological ordered space (X, τ_1, τ_2, R). It was a generalization of the study of general topological space, bitopological space and topological ordered space. Every bitopological ordered space (X, τ_1, τ_2, R) can be regarded as a bitopological space (X, τ_1, τ_2) if R is the equality relation "Δ".

* Corresponding author.
E-mail address: moona_hosny@yahoo.com (M. Hosny).
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Singal and Singal studied separation axioms PT_i-ordered spaces, $i = 0, 1, 2, 3, 4$ and PR_j-ordered spaces, $j = 2, 3$ in bitopological ordered spaces. After that time many authors have already been studied the bitopological ordered spaces [3,5–8].

Abo Elhamayel Abo Elwafa [3] introduced separation axioms P-completely normal ordered spaces, PT_5-ordered spaces and PR_j-ordered spaces, $j = 0, 1$ on the bitopological ordered spaces. Kandil et al. [7] studied the bitopological ordered spaces by using the supra-topological ordered spaces. They introduced new separations axioms P^*T_i-ordered spaces, $i = 0, 1, 2$ which was a generalization of previous one [2].

In 2015 Kandil et al. [1] used the concept of ideal $\mathcal{I}$ to introduce and study the ideal bitopological ordered spaces $(X, \tau_1, \tau_2, R, \mathcal{I})$. Clearly, if $\mathcal{I} = \{\phi\}$, then every ideal bitopological ordered space is bitopological ordered space. Therefore, these spaces are generalization of the bitopological ordered spaces and bitopological spaces. They used the notion of $\mathcal{I}$-increasing (decreasing) [9] sets and introduced separation axioms $\mathcal{I}PT_i$-ordered spaces, $(i = 0, 1, 2)$ in ideal bitopological ordered spaces.

The present paper is a continuation of [1]. So, the aim of the present paper was to study the separation axioms $\mathcal{I}PT_i$-ordered spaces, $i = 3, 4, 5$ and $\mathcal{I}PR_j$-ordered spaces, $j = 2, 3, 4$ on ideal bitopological ordered space $(X, \tau_1, \tau_2, R, \mathcal{I})$. The current separation axioms are based on the notion of $\mathcal{I}$-increasing (decreasing) sets. Comparisons between these axioms and the axioms in [2,3] have been obtained. The importance of the current study is that the new spaces are more general because the old one can be obtained from the current spaces when $\mathcal{I} = \{\phi\}$. Finally, we show that the properties of being $\mathcal{I}PT_i$-ordered spaces, $i = 3, 4, 5$ and $\mathcal{I}PR_j$-ordered spaces, $j = 2, 3, 4$ are preserved under a bijective, P-open and order (reverse) embedding mappings.

2. Preliminaries

Definition 2.1 [10,11] **.** A relation R on a non-empty set X is said to be:

1. reflexive if $(x, x) \in R$, for every $x \in X$,
2. symmetric if $(x, y) \in R \Rightarrow (y, x) \in R$, for every $x, y \in X$,
3. transitive if $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$, for every $x, y, z \in X$,
4. antisymmetric if $(x, y) \in R$ and $(y, x) \in R \Rightarrow x = y$, for every $x, y \in X$,
5. preorder relation if it is reflexive and transitive,
6. partial order relation if it is reflexive, antisymmetric and transitive, and the pair (X, R) is said to be a partially ordered set (or poset, for short).

Definition 2.2 [10] **.** For a non-empty set X and a partially order relation R on X, the pair (X, R) is said to be a partially ordered set (or poset, for short).

Definition 2.3 [12] **.** Let (X, R) be a poset. A set $A \subseteq X$ is said to be:

1. decreasing if for every $a \in A$ and $x \in X$, $xRa \Rightarrow x \in A$,
2. increasing if for every $a \in A$ and $x \in X$, $aRx \Rightarrow x \in A$.

Definition 2.4. A mapping $f : (X, R) \rightarrow (Y, R^*)$ is said to be:

1. increasing (decreasing) if for every $x_1, x_2 \in X, x_1Rx_2 \Rightarrow f(x_1)R^*f(x_2)(f(x_2)R^*f(x_1))$ [12],
2. order embedding if for every $x_1, x_2 \in X, x_1Rx_2 \Leftrightarrow f(x_1)R^*f(x_2)$ [13],
3. order reverse embedding if for every $x_1, x_2 \in X, x_1Rx_2 \Leftrightarrow f(x_2)R^*f(x_1)$ [3].

Definition 2.5 [14] **.** Let X be a non-empty set. A class τ of subsets of X is called a topology on X iff τ satisfies the following axioms:

1. $X, \phi \in \tau$,
2. arbitrary union of members of τ is in τ,
3. the intersection of any two sets in τ is in τ.

The members of τ are then called τ-open sets, or simply open sets. The pair (X, τ) is called a topological space. A subset A of a topological space (X, τ) is called a closed set if its complement A' is an open set.

Definition 2.6 [10] **.** Let (X, τ) be a topological space and $A \subseteq X$. Then $\tau\text{-}cl(A) = \cap\{F \subseteq X : A \subseteq F$ and F is closed$\}$ is called the τ-closure of a subset $A \subseteq X$

Definition 2.7 [4] **.** A bitopological space (bts, for short) is a triple (X, τ_1, τ_2), where τ_1 and τ_2 are arbitrary topologies for a set X.

Definition 2.8 [15,16] **.** A function $f : (X_1, \tau_1, \tau_2) \rightarrow (X_2, \eta_1, \eta_2)$ is said to be:

1. P.continuous (respectively P.open, P.closed) if $f : (X_1, \tau_i) \rightarrow (X_2, \tau_i), i = 1, 2$ are continuous (respectively open, closed).
2. P.homeomorphism if $f : (X_1, \tau_i) \rightarrow (X_2, \tau_i),\ i = 1, 2$ are homeomorphism.

Definition 2.9 [2] **.** A bitopological ordered space (bto-space, for short) has the form (X, τ_1, τ_2, R), where (X, R) is a poset and (X, τ_1, τ_2) is a bts.

The notion $a\overline{R}b$ means that a not related to b, i.e., $a\overline{R}b \Leftrightarrow (a, b) \notin R$.

Definition 2.10 [2] **.** A bto-space (X, τ_1, τ_2, R) is said to be:

1. Lower pairwise T_1 (LPT_1, for short)-ordered space if for every $a, b \in X$ such that $a\overline{R}b$, there exists an increasing τ_i-open set U contains a such that $b \notin U, i = 1$ or 2.
2. Upper pairwise T_1 (UPT_1, for short)-ordered space if for every $a, b \in X$ such that $a\overline{R}b$, there exists a decreasing τ_i-open set V contains b such that $a \notin V, i = 1$ or 2.
3. Pairwise T_1 (PT_1, for short), if it is LPT_1 and UPT_1-ordered space.

Definition 2.11 [2] **.** A bto-space (X, τ_1, τ_2, R) is said to be: