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Prediction and reconstruction of future and missing unobservable modified Weibull lifetime based on generalized order statistics

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Abstract When a system consisting of independent components of the same type, some appropriate actions may be done as soon as a portion of them have failed. It is, therefore, important to be able to predict later failure times from earlier ones. One of the well-known failure distributions commonly used to model component life, is the modified Weibull distribution (MWD). In this paper, two pivotal quantities are proposed to construct prediction intervals for future unobservable lifetimes based on generalized order statistics (gos) from MWD. Moreover, a pivotal quantity is developed to reconstruct missing observations at the beginning of experiment. Furthermore, Monte Carlo simulation studies are conducted and numerical computations are carried out to investigate the efficiency of presented results. Finally, two illustrative examples for real data sets are analyzed.

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1. Introduction

Prediction of future events (or reconstructing past events which have occurred but were unobservable) on the basis of past and present available information is one of the main problems in statistics. This problem has been extensively studied by many authors, including Lingappaiah [1], Aitchison and Dunsmore [2], Lawless [3,4], Kaminsky and Rhodin [5], Kaminsky and Nelson [6], Patel [7], Raqab et al. [8], Barakat et al. [9], El-Adll [10], El-Adll et al. [11], Barakat et al. [12] and AL-Hussaini et al. [13].

The ordered random variables without any doubt play an important role in such prediction problems. Since Kamps [14] had introduced the concept of gos as a unification of several models of ascendingly ordered random variables, the use of such concept has been steadily growing along the years. This is

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due to the fact that such concept includes important well-known models of ordered random variables that have been treated separately in the statistical literature. Kamps [14] defined gos first by defining uniform gos and then using the quantile transformation to obtain the $X(1,n,\tilde{m},k),\cdots,X(n,n,\tilde{m},k)$ based on cumulative distribution function (cdf) F. The joint probability density function (jpdf) of $X(1,n,\tilde{m},k),\cdots,X(n,n,\tilde{m},k)$ is given by

$$f^{X(1,n,\tilde{m},k),\ldots,X(n,n,\tilde{m},k)}(x_1,\ldots,x_n)$$
$$= k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}(1-F(x_i))^{m_i}f(x_i)\right)(1-f(x_n))^{k-1}f(x_n),$$

on the cone $F^{-1}(0)\le x_1\le\cdots\le x_n\le F^{-1}(1-)$ of $\mathbb{R}^n$. The model parameters are $n\in\mathbb{N}, n\ge 2$, $k>0, \tilde{m}=(m_1,\ldots,m_{n-1})\in\mathbb{R}^{n-1}$, $M_r=\sum_{j=r}^{n-1}m_j$, such that $\gamma_r=k+n-r+M_r>0$ for all $r\in\{1,\ldots,n-1\}$ and $\gamma_n=k$. Particular choices of the parameters $\gamma_1,\cdots,\gamma_n$ lead to different models, e.g., m-gos ($\gamma_n=k, \gamma_r=k+(n-r)(m+1), r=1,\cdots,n-1$), oos ($\gamma_n=1, \gamma_r=n-r+1, r=1,\ldots,n-1$, i.e., $k=1, m_i=0, i=1,\cdots,n-1$), sos ($\gamma_n=\alpha_n, \gamma_r=(n-r+1)\alpha_r, \alpha_r>0$, $r=1,\ldots,n-1$), pos with censoring scheme $(R_1,\ldots,R_M)$ ($\gamma_n=R_M+1, \gamma_r=n-r+1+\sum_{j=r}^{M}R_j$, if $r\le M-1$ and $\gamma_r=n-r+1+R_M$, if $r\ge M$) and upper records ($\gamma_r=1, 1\le r\le n$, i.e., $k=1, m_i=-1, i=1,\ldots,n-1$). Therefore, all the results obtained in the model of gos can be applied to the particular models choosing the respective parameters. For more details in the theory and applications of gos see Kamps [14], Ahsanullah [15], Kamps and Cramer [16], Cramer [17], Barakat et al. [9], El-Adll [18], Barakat [19], Atya [20] and Ahmad et al. [21].

Weibull distribution was originally introduced by the Swedish Waloddi Weibull (see Weibull [22]) which currently can be considered as one of the most important distributions in life testes and reliability engineering. Moreover, for more than 60 years Weibull distribution received increasing attention from several researchers in a wide variety of applications. Because of its various shapes of the probability density function and its convenient representation of the distribution/ survival function, the Weibull distribution has been used very effectively for analyzing lifetime data, particularly when the data are censored, which is very common in most life testing experiments. Moreover, Weibull distribution and its extensions are considered as the most important models in modern statistics because of its ability to fit data from various fields, ranging from life data to weather data or observations made in economics and business administration, in hydrology, in biology, and in the engineering sciences. Also, it has been used in many different areas such as material science, reliability engineering, physics, medicine, pharmacy economics, quality control, biology and other fields (for more details and applications of Weibull distribution see Rinne [23]).

Since 1958, the Weibull distribution has been modified by many researchers to allow for non-monotonic hazard functions. Lai et al. [24] proposed a three-parameter distribution known as MWD by multiplying the Weibull cumulative hazard function, αx^β, and $e^{\lambda x}$ which was later generalized to exponentiated form by Carrasco et al. [25]. Recent works of the modified Weibull include Sarhan and Zaindin [26], Sarhan and Apaloo, Atya [27,20] and Almalki and Nadarajah [28].

The pdf of the MWD is given by

$$f(x;\alpha,\lambda,\beta)=\begin{cases}\alpha(\beta+\lambda x)x^{\beta-1}e^{\lambda x}e^{-\alpha x^\beta e^{\lambda x}}, & x\ge 0;\\ 0, & x<0,\end{cases} \tag{1.1}$$

where α,β,λ are positive real numbers. The distribution function (cdf) is

$$F(x;\alpha,\lambda,\beta)=\begin{cases}0, & x<0;\\ 1-e^{-\alpha x^\beta e^{\lambda x}}, & x\ge 0.\end{cases} \tag{1.2}$$

The rest of this paper is organized as follows. In Section 2, the predictive pivotal quantities and their exact distributions are obtained. Section 3, includes simulation studies. Some applications for real data are presented in Section 4.

2. Pivotal quantities and their distributions

In this section, three pivotal quantities are proposed, two of them are used to construct prediction intervals for future observations from MWD based on gos, while the third is used to reconstruct missing observations. The cdf for each of the pivotal quantities is derived and then the limits of the predictive confidence interval are obtained. Furthermore, an approximate value of the expected upper limit for each predictive confidence interval is derived.

2.1. Prediction intervals of future observations

Suppose that $X(1,n,\tilde{m},k),\ldots,X(n,n,\tilde{m},k)$ are gos based on MWD with cdf given by (1.2). Define the following two pivotal quantities

$$P_1 := P_1(r,s,n,\tilde{m},k)=\frac{Y(s,n,\tilde{m},k)-Y(r,n,\tilde{m},k)}{Y(r,n,\tilde{m},k)}, \tag{2.1}$$

$$P_2 := P_2(r,s,n,\tilde{m},k)=\frac{Y(s,n,\tilde{m},k)-Y(r,n,\tilde{m},k)}{T_{r,n}}, \tag{2.2}$$

where

$$Y(i,n,\tilde{m},k)=\alpha(X(i,n,\tilde{m},k))^\beta e^{\lambda X(i,n,\tilde{m},k)}, \quad i=1,2,\ldots,n, \tag{2.3}$$
$$T_{r,n}=\sum_{i=1}^{r}\gamma_i(Y(i,n,\tilde{m},k)-Y(i-1,n,\tilde{m},k)), \quad \text{with}$$
$$Y(0,n,\tilde{m},k)=0. \tag{2.4}$$

The main aim of this subsection was to derive the exact distributions of P_1 and P_2 and to show that their distributions are free of the original distribution parameters, α, β and λ. The results are formulated in the following two theorems.

Theorem 2.1. *Suppose that $X(1,n,\tilde{m},k),\ldots,X(r,n,\tilde{m},k)$ are the first observed gos based on MWD with pdf (1.1) . Then the exact cdf of the pivotal quantity P_1, $F_{P_1}(p_1)$, is given by*

$$F_{P_1}(p_1)=1-C_{s-1}\sum_{i=r+1}^{s}\sum_{j=1}^{r}\frac{a_i^{(r)}(s)a_j(r)}{\gamma_i}(\gamma_j+\gamma p_1)^{-1}, \quad p_1\ge 0, \tag{2.5}$$

where,

$$C_{s-1}=\prod_{j=1}^{s}\gamma_j, a_i(r)=\prod_{j=1 j\ne i}^{r}\frac{1}{\gamma_{j,n}-\gamma_{i,n}}, 1\le i\le r\le n,$$

$$\text{and} a_i^{(r)}(s)=\prod_{j=r+1 j\ne i}^{s}\frac{1}{\gamma_{j,n}-\gamma_{i,n}}, r+1\le i\le s\le n.$$

Consequently, an observed $100(1-\delta)\%$ predictive confidence interval (PCI) for $X(s,n,\tilde{m},k), s>r$ is (ℓ,u_1), where $\ell=x_r$, and u_1 can be computed numerically from the relation