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Original Article

New ultraspherical wavelets collocation method for solving 2nth-order initial and boundary value problems

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Abstract In this paper, a new spectral algorithm based on employing ultraspherical wavelets along with the spectral collocation method is developed. The proposed algorithm is utilized to solve linear and nonlinear even-order initial and boundary value problems. This algorithm is supported by studying the convergence analysis of the used ultraspherical wavelets expansion. The principle idea for obtaining the proposed spectral numerical solutions for the above-mentioned problems is actually based on using wavelets collocation method to reduce the linear or nonlinear differential equations with their initial or boundary conditions into systems of linear or nonlinear algebraic equations in the unknown expansion coefficients. Some specific important problems such as Lane–Emden and Burger's type equations can be solved efficiently with the suggested algorithm. Some numerical examples are given for the sake of testing the efficiency and the applicability of the proposed algorithm.

Mathematics Subject Classification: 65M70; 65N35; 35C10; 42C10

1. Introduction

Many practical and physical problems in fields of science and engineering are formulated as boundary or initial value problems. The nonlinear boundary value problems are crucial in various applications as they arise frequently in many areas of science and engineering. For example, the deflection of a

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uniformly loaded rectangular plate supported over the entire surface by an elastic foundation and rigidly supported along the edges, satisfies a fourth-order differential equation see for example [6,8,12–15,20,28–30,41].

Spectral methods play prominent roles in solving various kinds of differential equations. It is known that there are three most widely used spectral methods, they are the tau, collocation, and Galerkin methods. Collocation methods have become increasingly popular for solving differential equations, in particular, they are very useful in providing highly accurate solutions to nonlinear differential equations (see, for example [1,6,8,12–15,19,20,28–30,41,49]).

High even-order differential equations have been extensively discussed by a large number of authors due to their great importance in various applications in many fields. For example, in the sequence of papers [18,16,17], the authors treated such equations by the Galerkin method. They constructed suitable basis functions which satisfy the boundary conditions of the given differential equation. For this purpose, they used compact combinations of various orthogonal polynomials. The suggested algorithms in these articles are suitable for handling one- and two-dimensional linear high even-order boundary value problems.

In this paper, we aim to give a new algorithm for handling both of linear and nonlinear even-order initial and boundary value problems based on employing ultraspherical wavelets collocation method (UWCM).

Wavelets cover a great area in mathematical models, and they have been used to handle a wide range of medical and engineering disciplines; in particular, wavelets are very appropriate in signal analysis, image segmentations, time frequency analysis and fast algorithms for easy implementation. Wavelets give accurate representation of a variety of functions and operators. Moreover, wavelets connect between fast numerical algorithms, (see [11,38,32]).

Legendre wavelets have been previously employed for solving various differential and integral equations (see for example, [35,34,36,46–48]). Also, first and second kinds Chebyshev wavelets have been used for solving some integer and fractional orders differential equations (see for example, [7,56,31,25]). Recently, Abd-Elhameed et al. in [2,3], have introduced new Chebyshev wavelets algorithms for solving second-order boundary value problems. Up to now, and to the best of our knowledge, the use of ultraspherical wavelets in various spectral numerical applications is traceless in the literature. This gives us a motivation to introduce and use the ultraspherical wavelets in various applications.

The outlines of this paper are organized as follows. In Section 2, we give some relevant properties of ultraspherical polynomials and their shifted forms. Moreover, in this section, ultraspherical wavelets are constructed. In Section 3, the convergence of the ultraspherical wavelets expansion is proved. Section 4 is concerned with presenting and implementing a numerical algorithm for solving even-order linear and nonlinear initial or boundary value problems based on employing ultraspherical wavelets together with the spectral collocation method. Section 5 is concerned with considering some numerical examples aiming to illustrate the efficiency and applicability of the developed algorithm. Conclusions are given in Section 6.

2. Some properties of ultraspherical polynomials and ultraspherical wavelets

This section is concerned with presenting an overview on ultraspherical polynomials and their shifted polynomials. Also, ultraspherical wavelets are constructed in this section.

2.1. *Some properties of ultraspherical polynomials*

The ultraspherical polynomials are a class of symmetric Jacobi polynomials. They are a sequence of orthogonal polynomials defined on the interval $(-1,1)$ with respect to the weight function $w(x) = (1-x^2)^{\alpha-\frac{1}{2}}$ associated with the real parameter $(\alpha > -\frac{1}{2})$. Explicitly, they satisfy the orthogonality property

$$\int_{-1}^{1} (1-x^2)^{\alpha-\frac{1}{2}}\, C_m^{(\alpha)}(x) C_n^{(\alpha)}(x)\, dx = \begin{cases} \frac{\pi\, 2^{1-2\alpha}\, \Gamma(n+2\alpha)}{n!\,(n+\alpha)\,(\Gamma(\alpha))^2}, & m = n, \\ 0, & m \neq n. \end{cases} \tag{1}$$

Also, they are the eigenfunctions of the following singular Sturm–Liouville equation

$$(1-x^2)\, D^2\phi_m(x) - (2\alpha+1)x\, D\phi_m(x) + m(m+2\alpha)\, \phi_m(x) = 0,$$

$$D \equiv \frac{d}{dx}.$$

For more properties of ultraspherical polynomials, one can be referred to [5].

The following integral formula (see, [5]) is needed

$$\int C_n^{(\alpha)}(x)\, w(x)\, dx = -\frac{2\alpha\,(1-x^2)^{\alpha+\frac{1}{2}}}{n(n+2\alpha)} C_{n-1}^{(\alpha+1)}(x), \quad n \geq 1. \tag{2}$$

Also, the following theorem is useful in deriving the convergence theorem for the expansion of the ultraspherical wavelets.

Theorem 1 ([22] Bernstein-type inequality). *The following inequality holds for ultraspherical polynomials:*

$$(\sin\theta)^{\alpha}\, |C_n^{(\alpha)}(\cos\theta)| < \frac{2^{1-\alpha}\,\Gamma\left(n+\frac{3\alpha}{2}\right)}{\Gamma(\alpha)\,\Gamma\left(n+1+\frac{\alpha}{2}\right)}, \quad 0 \leq \theta \leq \pi, \ \ 0 < \alpha < 1. \tag{3}$$

2.2. *Shifted ultraspherical polynomials*

The shifted ultraspherical polynomials are defined on [0, 1] by $\overset{\sim(\alpha)}{C_n}(x) = C_n^{(\alpha)}(2x-1)$. All results of ultraspherical polynomials can be easily transformed to give the corresponding results for their shifted polynomials. The orthogonality relation for $\overset{\sim(\alpha)}{C_n}(x)$ with respect to the weight function $\tilde{\omega} = (x-x^2)^{\alpha-\frac{1}{2}}$ is given by

$$\int_{0}^{1} (x-x^2)^{\alpha-\frac{1}{2}}\, \overset{\sim(\alpha)}{C_n}(x)\, \overset{\sim(\alpha)}{C_m}(x)\, dx = \begin{cases} \dfrac{\pi\, 2^{1-4\alpha}\, \Gamma(n+2\alpha)}{n!\,(n+\alpha)\,(\Gamma(\alpha))^2}, & m = n, \\ 0, & m \neq n. \end{cases}$$

For more properties of ultraspherical polynomials see [40].

2.3. *Construction of ultraspherical wavelets*

Wavelets constitute of a family of functions constructed from dilation and translation of single function called the mother