Journal of the Egyptian Mathematical Society (2016) 24, 1–4

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

Cleanness of overrings of polynomial rings

Hani A. Khashan *, **Worood Burhan**

Department of Mathematics, Al al-Bayt University, Mafraq, Jordan

Received 22 February 2014; revised 17 June 2014; accepted 2 August 2014
Available online 11 February 2015

KEYWORDS

Clean rings;
Almost clean rings;
The rings $R(x)$ and $R\langle x\rangle$

Abstract Let R be a commutative ring with identity. R is called clean (respectively, almost clean) if every element in R is a sum of an idempotent and a unit (respectively, a regular element). In this paper, we clarify conditions under which the two overrings $R(x)$ and $R\langle x\rangle$ of the ring of polynomials $R[x]$ are (almost) clean rings.

2000 MATHEMATICS SUBJECT CLASSIFICATION: 13A15; 13A99; 13B99

1. Introduction

Throughout this paper a ring R will be commutative with identity, $U(R)$ denotes the set of all units of $R, reg(R)$ the set of regular elements of $R, Z(R)$ the set of all zero divisors of R and $Id(R)$ the set of all idempotent elements of R. If $f\in R[x]$, the ring of all polynomials over R, then $C(f)$ denotes the ideal of R generated by the coefficients of f. Let $S=\{f\in R[x]:C(f)=R\}$ and $W=\{f\in R[x]:f$ is monic$\}$. Then S and W are regular multiplicatively closed subsets of $R[x]$ and the rings of fractions $S^{-1}R[x]$ and $W^{-1}R[x]$ are denoted by $R(x)$ and $R\langle x\rangle$ respectively. As clearly $W\subseteq S$, then $R(x)$ is itself a ring of fractions of $R\langle x\rangle$, see Theorem 3.16 in [1]. Some basic properties and related theorems of $R(x)$ and $R\langle x\rangle$ can be found in [2–5] where we can see that the rings $R, R(x)$ and $R\langle x\rangle$ share many properties. In the following lemma, we survey some properties of the rings $R[x], R(x)$ and $R\langle x\rangle$ that will be needed in this paper.

Lemma 1.1. *Let R be a ring. Then*

(1) *If f and g are two nonzero polynomials in $R[x]$ with $deg(f)=k$, then $c(g)^{k+1}c(f)=c(g)^{k}c(fg)$.*
(2) *If R is a finite dimensional ring, then $dim(R(x))=dim(R\langle x\rangle)=dim(R[x])-1$.*
(3) *$Id(R(x))=Id(R\langle x\rangle)=Id(R)$.*
(4) *$U(R(x))=\left\{\frac{f}{g}\in R(x):c(f)=c(g)=R\right\}$.*
(5) *There is a one to one correspondence between the maximal (minimal) ideals of R and the maximal (minimal) ideals of $R(x)$ (or $R\langle x\rangle$) given by, $P\leftrightarrow PR(x)$ (or $P\leftrightarrow PR\langle x\rangle$).*
(6) *If I is an ideal of R, then $IR(x)\cap R=IR\langle x\rangle\cap R=I$.*
(7) *If $R=R_1\times R_2\times\cdots\times R_n$ is a direct product of rings, then $R(x)=R_1(x)\times R_2(x)\times\cdots\times R_n(x)$ and $R\langle x\rangle=R_1\langle x\rangle\times R_2\langle x\rangle\times\cdots\times R_n\langle x\rangle$.*
(8) *If R is an indecomposable ring, then $R(x)$ and $R\langle x\rangle$ are also indecomposable.*
(9) *If R is a Noetherian ring, then $R(x)$ and $R\langle x\rangle$ are also Noetherian.*

* Corresponding author.
E-mail address: hakhashan@aabu.edu.jo (H.A. Khashan).

Peer review under responsibility of Egyptian Mathematical Society.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.joems.2014.08.003
1110-256X

Proof. See [2,5,4]. □

Let R be a ring. An element $r \in R$ is called clean (respectively, almost clean) if $r = u + e$ where $u \in U(R)$ (respectively, $u \in reg(R)$) and $e \in Id(R)$. If every element in R is clean (respectively, almost clean), then R is called a clean (respectively, almost clean) ring. Since clearly every unit in a ring R is regular, then the class of almost clean rings is a generalization of the class of clean rings. This generalization is proper since any non quasi-local integral domain is almost clean which is not clean, see [6]. The class of clean rings (not necessarily commutative) was first introduced by Nicholson [7] as early as 1977 in his study of lifting idempotents and exchange rings. In fact, he proved that the class of commutative clean rings coincide with the class of commutative exchange rings. Since then, a great deal is known about clean rings and their generalizations. In particular in 2002, Anderson and Camillo [6] studied commutative clean rings and gave a characterization for a Noetherian clean rings. Then Ahn and Anderson [8] in 2006, defined and studied weakly clean and almost clean rings as two important generalizations of clean rings. In 2012, Anderson and Badawi [9] gave a complete classification of the clean elements of the ring $R[x]$. More results about clean rings and some of their generalizations can be found also in [10,11].

In this paper, we clarify conditions under which the following statements are equivalent for a ring R:

(1) R is clean (almost clean).
(2) $R\langle x\rangle$ is clean (almost clean).
(3) $R(x)$ is clean (almost clean).

2. When $R(x)$ and $R\langle x\rangle$ are (almost) clean rings

In this section, we clarify conditions on a ring R under which the ring $R(x)$ and $R\langle x\rangle$ are clean rings or almost clean rings.

Theorem 2.1. *Let R be a ring and consider the following statements*

(1) *R is clean.*
(2) *$R\langle x\rangle$ is clean.*
(3) *$R(x)$ is clean.*

then we have $(3) \Rightarrow (1)$ *and* $(2) \Rightarrow (1)$.

Proof. $(3) \Rightarrow (1)$: Suppose that $R(x)$ is a clean ring and let $r \in R \subseteq R(x)$. Since $R(x)$ is clean, then there exists $\frac{f}{g} \in U(R(x))$ and $\frac{f'}{g'} \in Id(R(x))$ such that $r = \frac{f}{g} + \frac{f'}{g'}$. Hence by Lemma 1.1, we have $c(g) = c(f) = R$ and $\frac{f'}{g'} = e \in Id(R)$. Therefore, $\frac{f}{g} = r - e \in U(R(x)) \cap R$. It is enough now to prove that $U(R(x)) \cap R = U(R)$. Let $\frac{f}{g} = r \in U(R(x)) \cap R$, then $f = rg$ where $c(g) = c(f) = R$ and so again by Lemma 1.1, $c(g)^1 c(r) = c(g)^0 c(rg) = c(rg)$. Hence, $(r) = c(r) = c(rg) = c(f) = R$ and so $\frac{f}{g} = r \in U(R)$. The other containment is obvious. Therefore, R is a clean ring.

$(2) \Rightarrow (1)$: As $R \subseteq R\langle x\rangle \subseteq R(x)$, we have $U(R) \subseteq U(R\langle x\rangle) \cap R \subseteq U(R(x)) \cap R \subseteq U(R)$ and so $U(R\langle x\rangle) \cap R = U(R)$. Since also $Id(R) = Id(R\langle x\rangle)$, then similar to the proof of $(3) \Rightarrow (1)$, we see that R is a clean ring. □

Let R be a ring and let $\widehat{P}$ be a prime ideal of $R(x)$. Then $\widehat{P} = S^{-1}Q$ where Q is a prime ideal of $R[x]$ and so $\widehat{P} \cap R[x] = S^{-1}Q \cap R[x] = Q$. Therefore, $\widehat{P} \cap R = Q \cap R$ is a prime ideal of R. Similarly, $\breve{P} \cap R$ is a prime ideal of R for any prime ideal $\breve{P}$ of $R\langle x\rangle$.

We recall that a ring R is called a pm-ring if each prime ideal of R is contained in a unique maximal ideal. In the following lemma, we clarify a condition under which $R(x)$ and $R\langle x\rangle$ are pm-rings.

Lemma 2.2. *Let R be a ring. Then the following are equivalent*

(1) *R is a pm-ring.*
(2) *$R(x)$ is a pm-ring.*
(3) *$R\langle x\rangle$ is a pm-ring.*

Proof. $(1) \Rightarrow (2)$: Suppose that R is a pm-ring and let $\widehat{P}$ be a prime ideal of $R(x)$ and suppose that there are two distinct maximal ideals $\widehat{M}_1$ and $\widehat{M}_2$ of $R(x)$ such that $\widehat{P} \subseteq \widehat{M}_1$ and $\widehat{P} \subseteq \widehat{M}_2$. By Lemma 1.1, $\widehat{M}_1 = M_1R(x)$ and $\widehat{M}_2 = M_2R(x)$ for some distinct maximal ideals M_1 and M_2 of R. Thus, $\widehat{P} \cap R \subseteq M_1R(x) \cap R = M_1$ and $\widehat{P} \cap R \subseteq M_2R(x) \cap R = M_2$. But, $\widehat{P} \cap R$ is a prime ideal of R which contradicts that R is a pm-ring. Therefore, any prime ideal of $R(x)$ is contained in a unique maximal ideal and so $R(x)$ is a pm-ring.

$(2) \Rightarrow (3)$: Suppose $R(x)$ is a pm-ring and let $\breve{P}$ be a prime ideal of $R\langle x\rangle$. As $R(x) = T^{-1}R\langle x\rangle$ for some multiplicatively closed subset T of $R\langle x\rangle$, then $T^{-1}\breve{P}$ is a prime ideal of $R(x)$. Thus, there is a unique maximal ideal $\widehat{M}$ of $R(x)$ such that $T^{-1}\breve{P} \subseteq \widehat{M}$. Write $\widehat{M} = T^{-1}Q$ where Q is a prime ideal of $R\langle x\rangle$. Then it follows clearly that $\widehat{M} \cap R\langle x\rangle = T^{-1}Q \cap R\langle x\rangle = Q$ is the unique maximal ideal of $R\langle x\rangle$ containing $\breve{P}$. Therefore, $R\langle x\rangle$ is a pm-ring.

$(3) \Rightarrow (1)$: Suppose $R\langle x\rangle$ is a pm-ring. Let P be a prime ideal of R and suppose that there are two distinct maximal ideals M_1 and M_2 of R such that $P \subseteq M_1$ and $P \subseteq M_2$. Then $PR\langle x\rangle$ is a prime ideal of $R\langle x\rangle$ with $PR\langle x\rangle \subseteq M_1R\langle x\rangle$ and $PR\langle x\rangle \subseteq M_2R\langle x\rangle$. But, $M_1R\langle x\rangle$ and $M_2R\langle x\rangle$ are distinct maximal ideals of $R\langle x\rangle$ which is a contradiction. Thus, R is a pm-ring. □

In [6], it has been proved that every clean ring is a pm-ring. The converse is true under a certain condition as we can see in the following Lemma.

Lemma 2.3 [6]. *Let R be a ring with a finite number of minimal prime ideals. Then the following are equivalent:*

(1) *R is a finite direct product of quasi-local rings.*
(2) *R is a clean ring.*
(3) *R is a pm-ring.*