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On Solving Ill Conditioned Linear Systems

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Abstract

This paper presents the first results to combine two theoretically sound methods (spectral projection and multigrid methods) together to attack ill conditioned linear systems. Our preliminary results show that the proposed algorithm applied to a Krylov subspace method takes much fewer iterations for solving an ill conditioned problem downloaded from a popular online sparse matrix collection.

Keywords: numerical analysis, scientific computing, multigrid, nonlinear data driven applications

1 Introduction

It is well known that robustness and efficiency of iterative methods are affected by the condition number of a linear system. When a linear system has a large condition number, usually due to eigenvalues that are close to the origin of the spectrum domain, iterative methods tend to take many iterations before a convergence criterion is satisfied. Sometimes, iterative methods will fail to converge within a reasonable computer elapsed time, or even do not converge at all, if the condition number is too large. Unstable linear systems, or systems with large condition numbers, are called ill conditioned. For an ill conditioned linear system, slight changes in the coefficient matrix or the right-hand-side cause large changes in the solution. Typically, roundoff error in the computer arithmetics can cause instability when attempts are made to solve an ill conditioned system either directly or iteratively on a computer.

It is widely recognized that linear systems resulting from discretizing ill posed integral equations of the first kind are highly ill conditioned. This is because the eigenvalues for the first kind integral equations with continuous or weakly singular kernels have an accumulation point at zero. Integral equations of the first kind are frequently seen in statistics, such as unbiased estimation, estimating a prior distribution on a parameter given the marginal distribution of the data and the likelihood, and similar tests for normal theory problems. They also arise from indirect measurements and nondestructive testing in inverse problems. Other ill conditioned linear systems can be seen in training of neural networks, seismic analysis, Cauchy problem for

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parabolic equations, and multiphase flow of chemicals. For pertinent references of ill conditioned linear systems, see Engl [17] and Groetsch [22].

Solving these ill conditioned linear algebra problems becomes a long standing bottleneck for advancing the use of iterative methods. The convergence of iterative methods for ill conditioned problems, however, can be improved by using preconditioning. Development of preconditioning techniques is therefore a very active research area. A preconditioning strategy that *deflates* few isolated external eigenvalues was first introduced by Nicolaides [27], and investigated by several others [26, 36, 39, 20]. The deflation strategy is an action that removes the influence of a subspace of the eigenspace on the iterative process. A common way to deflate an eigenspace is to construct a proper projector P as a preconditioner and solve

$$PAx = Pb, \quad P, A \in \mathbb{C}^{N\times N}. \tag{1}$$

The deflation projector P, which is orthogonal to the matrix A and the vector b against some subspace, is defined by

$$P = I - AZ(Z^H AZ)^{-1}Z^H, \quad Z \in \mathbb{C}^{N\times m}, \tag{2}$$

where Z is a matrix of deflation subspace, i.e., the space to be projected out of the residual, and I is the identity matrix of appropriate size [30, 20]. We assume that (1) $m \ll N$ and (2) Z has rank m. A deflated $N\times N$ system (1) has an eigensystem different from that of $Ax = b$. Suppose that A is diagonalizable, and set $Z = [v_1, \cdots, v_m]$, whose columns are eigenvectors of A associated with eigenvalues $\lambda_1, \cdots, \lambda_m$. Then the spectrum $\sigma(PA)$ would contain the same eigenvalues of A, except $\lambda_1, \cdots, \lambda_m$. Usually, eigenvectors are not easily available. This motivates us to develop an efficient and robust algorithm for finding an approximate deflation subspace, other than using the exact eigenvectors to construct the deflation projector P.

Suppose that we want to deflate a set of eigenvalues of A enclosed in a circle Γ that is centered at the origin with the radius r. Without loss of generality, let this set of eigenvalues be $\{\lambda_1, \cdots, \lambda_k\}$. Let the subspace spanned by the corresponding eigenvectors of $\{\lambda_1, \cdots, \lambda_k\}$ be $\mathcal{Z}_k = \mathrm{Span}\{v_1, \cdots, v_k\}$. Then the deflation subspace matrix Z in (2) obtained by randomly selecting m vectors from $\mathcal{Z}_k$ can be written as a contour integral [31]

$$Z = \frac{1}{2\pi\sqrt{-1}} \oint_\Gamma (zI - A)^{-1} Y \, dz, \tag{3}$$

where Y is a random matrix of size $N\times m$. If the above contour integral is approximated by a Gaussian quadrature, we have

$$Z = \sum_{i=1}^{q} \omega_i (z_i I - A)^{-1} Y, \tag{4}$$

where ω_i are the weights, z_i are the Gaussian points, and q is the number of Gaussian points on Γ for the quadrature. It is worth noting that (4) is required to solve q shifted linear systems $(z_i I - A)X = Y$, $i = 1, \cdots, q$. Using (4) for the deflation projector P in (2), the preconditioned linear system (1) is no longer severely ill conditioned.

We remark that the construction of a deflation subspace matrix Z through (4) is motivated by the works in [33, 34, 28, 37].

2 Methodology

We consider the solution of the linear system

$$Ax = b \tag{5}$$