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Application of asymptotic analysis to the two-scale modeling of small defects in mechanical structures

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ABSTRACT

This work aims at designing a numerical strategy towards assessing the nocivity of a small defect in terms of its size and position in a structure, at low computational cost, using only a mesh of the defect-free reference structure. The modification of the fields induced by the presence of a small defect is taken into account by using asymptotic corrections of displacements or stresses. This approach helps determining the potential criticality of defects by considering trial micro-defects with varying positions, sizes and mechanical properties, taking advantage of the fact that parametric studies on defect characteristics become feasible at virtually no extra computational cost. The proposed treatment is validated and demonstrated on two numerical examples involving 2D elastic configurations.

1. Introduction

The role played by defects in the onset and development of rupture is crucial and has to be taken into account in order to assess the potential failure of mechanical structures. Difficulties in this context include (i) the length scale of defects often being much smaller than the structure length scale, and (ii) the frequent randomness of the location, nature and geometry of defects. Even with deterministic approaches, taking such defects into consideration by standard methods entails geometrical discretizations at the defect scale, leading to costly computations and hindering parametric studies for varying defect location and characteristics.

We address situations that require modeling a single small flaw, or a moderate number of such flaws, and therefore do not pertain to homogenization. Such isolated defects are usually either omitted (if small enough) or fully modelled. In the former case, initiation and eventual propagation of cracks leading to failure may be missed, while the latter case both complicates finite element (FE) model preparation and significantly increases computational costs due to severe mesh refinement in the region surrounding a modelled flaw.

In this work, we propose to address the latter issues by resorting to an efficient two-scale numerical strategy which can accurately predict the mechanical state perturbation caused by isolated inhomogeneities embedded in an elastic (background) material, without directly modeling them. To ensure computational efficiency, the analysis uses only a FE mesh for the defect-free structure, whose mesh size is hence not influenced by the (small) defect scale. The latter is instead taken into account by means of an asymptotic expansion, as previously done in Dambrine and Vial (2007), Brancherie et al. (2008) and Bonnaillie-Noël et al. (2010) for modeling surface-breaking void defects (see also Silva et al., 2011 where the concept of topological derivative (Novotny and Sokołowski, 2012; Bonnet and Delgado, 2013; Bonnet and Cornaggia, 2017) is used for predicting the eventual nocivity of surface-breaking small cracks). Here we are addressing the case of a small internal inhomogeneity (or a finite number thereof) embedded in an elastic solid. This includes traction-free voids as a special case, thus covering (small) objects variously referred to in the literature (see e.g. Mery et al., 2002; Wang and Liao, 2002; Mura, 1987) as inhomogeneities, heterogeneities, cracks, holes, porosities, inclusions... We rely on existing results on small-inhomogeneity asymptotics for elastic solids (Ammari et al., 2002; Ammari and Kang, 2007; Ammari et al., 2013; Nazarov et al., 2010; Bonnet and Delgado, 2013; Bonnet and Cornaggia, 2017), which prominently involve *elastic moment tensors* (EMTs) associated with elastic inhomogeneities (Ammari et al., 2002; Ammari and Kang, 2007; Bonnet and Delgado, 2013), and combine them into a simple computational treatment, whose capabilities (prominently among them the ability to conduct inexpensive parametric studies) are then demonstrated on two examples.

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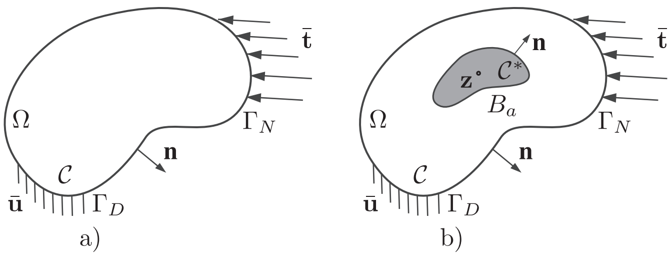

Fig. 1. Reference (a) and perturbed (b) solids. The inhomogeneity B_a located at $\mathbf{z}$ is the shaded subdomain in (b).

The paper is organized as follows. After defining the relevant background and transmission problems (Section 2), the small-inhomogeneity asymptotic expansion in terms of the displacement perturbation is introduced, focusing on the far field, in Section 3. Therein, the key ingredients for its evaluation (elastostatic Green's tensor and EMT) are surveyed, and the resulting proposed computational treatment is given. Two validation and demonstration examples are then presented in Section 4. Section 5 closes the paper with concluding remarks and directions for future work.

2. Problem definition

We consider a linearly elastic body occupying a bounded domain $\Omega \subset \mathbb{R}^d$ (where $d = 2$ or 3 is the spatial dimensionality), whose boundary Γ is partitioned as $\Gamma = \Gamma_D \cup \Gamma_N$, with $\Gamma_D \cap \Gamma_N = \emptyset$ to ensure well-posedness of boundary value problems. The parts Γ_D and Γ_N respectively support a prescribed traction $\bar{\mathbf{t}}$ and a prescribed displacement $\bar{\mathbf{u}}$, while a body force density $\mathbf{f}$ is applied in Ω. These boundary conditions are chosen for definiteness, and any other set of well-posed boundary conditions could be chosen instead with minimal changes. On the basis of this fixed geometrical and loading configuration, we consider two situations, namely (i) a reference solid characterized by a given elasticity tensor $\mathcal{C}$, which defines the background solution, and (ii) a perturbed solid constituted of the same background material except for a small inhomogeneity whose material is characterized by the elasticity tensor $\mathcal{C}^\star$. The aim of this work is to formulate a computational approach allowing to treat case (ii) as a perturbation of the background solution (i), in particular avoiding any meshing at the small inhomogeneity scale. This will be achieved by applying known results on the asymptotic expansion of the displacement perturbation with respect to the small characteristic size a of the inhomogeneity to case (ii).

2.1. Background solution (case (i))

The background solution in terms of displacement field $\mathbf{u}$ arising in the reference solid Ω with elasticity tensor $\mathcal{C}$ (Fig. 1a) due to prescribed excitation $(\mathbf{f}, \bar{\mathbf{t}}, \bar{\mathbf{u}})$, corresponding to case (i) above, solves the problem

$$\operatorname{div}(\mathcal{C} : \boldsymbol{\varepsilon}[\mathbf{u}]) + \mathbf{f} = \mathbf{0} \text{ in } \Omega, \quad \mathbf{t}[\mathbf{u}] = \bar{\mathbf{t}} \text{ on } \Gamma_N, \quad \mathbf{u} = \bar{\mathbf{u}} \text{ on } \Gamma_D, \qquad (1)$$

where the linearized strain tensor $\boldsymbol{\varepsilon}[\mathbf{w}]$ and the traction vector $\mathbf{t}[\mathbf{w}]$ associated with a given displacement $\mathbf{w}$ are given by

$$\text{(a)} \quad \boldsymbol{\varepsilon}[\mathbf{w}] = (\nabla \mathbf{w} + \nabla^T \mathbf{w})/2, \quad \text{(b)} \quad \mathbf{t}[\mathbf{w}] = (\mathcal{C} : \boldsymbol{\varepsilon}[\mathbf{w}]) \cdot \mathbf{n}, \qquad (2)$$

with $\mathbf{n}$ denoting the unit outward normal to Γ. In (2 b) and hereinafter, symbols '$\cdot$' and '$:$' denote single and double inner products.

2.2. Transmission problem for a small inhomogeneity (case (ii))

The elastic body occupies the same domain Ω but now contains a small defect, in the form of an inhomogeneity located at $\mathbf{z} \in \Omega$, embedded in the background material (Fig. 1b). The inhomogeneity occupies the domain $B_a := \mathbf{z} + a\mathcal{B}$, where the smooth fixed domain $\mathcal{B} \subset \mathbb{R}^d$ centered at the origin defines the defect shape, and has elastic properties described by the tensor $\mathcal{C}^\star$. The inhomogeneous elastic properties of the whole perturbed solid are therefore defined as

$$\mathcal{C}_a := \mathcal{C} + \Delta\mathcal{C}\,\chi_{B_a}, \qquad (3)$$

where χ_D is the characteristic function of a domain D and $\Delta\mathcal{C} := \mathcal{C}^\star - \mathcal{C}$ denotes the elasticity tensor perturbation.

The displacement field $\mathbf{u}_a$ arising in the solid containing the small inhomogeneity B_a due to the same prescribed excitation $(\mathbf{f}, \bar{\mathbf{t}}, \bar{\mathbf{u}})$, solves the transmission problem

$$\begin{aligned} &\operatorname{div}(\mathcal{C}_a : \boldsymbol{\varepsilon}[\mathbf{u}_a]) + \mathbf{f} = \mathbf{0} \text{ in } B_a \cup (\Omega \setminus \overline{B_a}), \quad \mathbf{t}[\mathbf{u}_a] = \bar{\mathbf{t}} \text{ on } \Gamma_N, \\ &\mathbf{u}_a = \bar{\mathbf{u}} \text{ on } \Gamma_D, \mathbf{u}_a|_- = \mathbf{u}_a|_+ \quad \text{and} \quad \mathbf{t}^\star[\mathbf{u}_a]\,|_- = \mathbf{t}[\mathbf{u}_a]\,|_+ \text{ on } \partial B_a, \end{aligned} \qquad (4)$$

where the traction operator $\mathbf{t}^\star$ is defined by (2 b) with $\mathcal{C}$ replaced by $\mathcal{C}^\star$ and the $\pm$ subscripts indicate traces relative to B_a and $\Omega \setminus \overline{B_a}$, respectively.

3. Computation of small-inhomogeneity solution asymptotics

This section develops our proposed methodology. The small-inhomogeneity asymptotic expansion in terms of the displacement perturbation is introduced, focusing on the far field, in Section 3.1. The key ingredients for its evaluation, namely the elastostatic Green's tensor and elastic moment tensors, are surveyed in Sections 3.2 and 3.3, respectively, the resulting proposed computational treatment being then given in Section 3.4. Some useful explicit formulas for the plane strain case are finally gathered in Section 3.5.

3.1. Asymptotic approximation of displacement perturbation

We begin by introducing the displacement perturbation

$$\mathbf{v}_a := \mathbf{u}_a - \mathbf{u}, \qquad (5)$$

where $\mathbf{u}_a$ and $\mathbf{u}$ solve problems (4) and (1), respectively corresponding to the perturbed and background configurations. An asymptotic analysis of $\mathbf{v}_a$ with respect to the characteristic defect size a provides a way to evaluate the influence of the location, size, shape and material characteristics of defects on the solution $\mathbf{u}_a$. Available asymptotic approximations, such as those used in this work, nearly always rely on a constitutive linearity assumption (here, linear elasticity), with the notable exception of Amstutz and Bonnafé (2017).

Two kinds of asymptotic expansions of $\mathbf{v}_a$ may be defined, namely *inner* and *outer* expansions (Maz'ya et al., 2000). They focus on the two scales involved: (a) the structure scale, where points are described using "ordinary" coordinates $\mathbf{x} \in \Omega$, and (b) the defect scale corresponding to the characteristic length a of the inhomogeneity, with rescaled coordinates $\bar{\mathbf{x}} := (\mathbf{x} - \mathbf{z})/a$. This description is directly related to the slow and fast variables used in Dambrine and Vial (2007) and Brancherie et al. (2008).

Inner expansion. The inner expansion has the form (Beretta et al., 2011; Bonnet and Delgado, 2013, Prop. 3.2)

$$\mathbf{v}_a(\mathbf{x}) = a\mathbf{v}_{\mathcal{B}}[\boldsymbol{E}](\bar{\mathbf{x}}) + o(a), \qquad \boldsymbol{E} = \nabla \mathbf{u}(\mathbf{z}), \qquad (6)$$

having set $\mathbf{v}_{\mathcal{B}}[\boldsymbol{E}] := \mathbf{u}_{\mathcal{B}}[\boldsymbol{E}] - \boldsymbol{E} \cdot \bar{\mathbf{x}}$ in terms of the solution $\mathbf{u}_{\mathcal{B}}[\boldsymbol{E}]$ of the *free-space transmission problem* (FSTP)