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# A variational formulation for thermomechanically coupled low cycle fatigue at finite strains

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ABSTRACT

In this paper, a constitutive model suitable for the analysis of Low Cycle Thermo-Mechanical Fatigue in metals is elaborated. The model is based on finite strain elastoplasticity coupled to continuum damage theory. It is embedded into a thermodynamical framework allowing to consistently capture the interplay between mechanics and thermal effects. It is shown that the fully coupled constitutive model can be re-written into a variationally consistent manner such that all (state) variables follow jointly and naturally from minimizing an incrementally defined functional. By discretizing this time-continuous functional in time by means of implicit integration schemes a numerically efficient implementation is proposed. In order to predict the temperature increase caused by plastic deformations realistically, the pre-loading history of the considered specimen is accounted for by non-zero initial internal variables. A comparison of the results predicted by the novel constitutive model to those corresponding to experiments (Ultimet alloy) shows that the predictive capabilities of the final model are excellent.



## 1. Introduction

Failure induced by an interplay between cyclic mechanical and cyclic thermal load is referred to as Thermo-Mechanical Fatigue (TMF). This type of failure/fatigue is relevant for a broad range of different engineering applications. Turbine blades in aircraft engines or blades in gas turbines are two representative examples. In line with classical fatigue (only cyclic mechanical loading), TMF can be subdivided into different classes depending on the number of loading cycles defining the lifetime of the considered structure. Within this paper, a constitutive model for Low Cycle TMF in metals is proposed (LCTMF). In this case, total failure is observed already for a relatively small number of loading cycles $N_f$ (typically, $10^2 < N_f < 10^4$).

Certainly, a classification of fatigue simply by means of the number of loading cycles up to failure is not always meaningful. To be more precise, a meaningful characterization of the different thermo-mechanical fatigue mechanisms is only possible by means of a careful material characterization.

Focusing on thermoelasticity and monotonic loading, a careful material characterization can be found in Oliferuk et al. (2012), cf. Boulanger et al. (2004). Concerning the temperature increase resulting from dissipation due to plastic deformations for monotonic loading, experiments and the respective results can be found in Rosakis et al. (2000) and Hodowany et al. (2000). For more recent works, the reader is referred to Baig et al. (2013) and Xu and Huang (2013) and reference cited therein. While the citations given so far are related to monotonic loading, the impact of cyclic loading (low cycle fatigue) is experimentally studied in Jiang et al. (2004), Harvey and Bonenberger (2000), Naderi and Khonsari (2010) and Amiri and Khonsari (2010). Without going too much into detail, these experiments clearly show that the number of loading cycles is only an indicator for the classification of the different fatigue mechanisms. The second important indicator for LCF in metals is the occurrence of plastic deformations. Consequently, a physically sound model suitable for the analysis of LCTMF in metals has to capture (1) the interplay between elastoplastic deformations and material degradation (material damage) and (2) the interplay between mechanical and thermal effects.

With respect to the interplay between elastoplastic deformations and material degradation, a broad variety of different models can be found in the literature. Since the final applications of the LCTMF model presented in this paper are at the macroscale, only macroscopic models are considered here. Such models can be either micromechanically motivated (Gurson, 1977; Rousselier, 1981; Tvergaard and Needleman, 1984; Chabanet et al., 2003) or they approximate the respective damage processes by means of the general formalism of internal variables, cf. Kachanov (1958),

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Cordebois and Sidoroff (1979), Lemaitre (1992) and Lemaitre and Desmorat (2005). Within the aforementioned first class of models, effective evolution equations are derived by analyzing representative volume elements with evolving defects. Although this idea is certainly physically sound, a rigorous analytical homogenization of the RVE resulting in the macroscopic evolution equations is usually not possible due to the complexity of the underlying model within the RVE. Since the phenomena at the micro-scale which define the macroscopic LCTMF response are indeed complex, the second class of modeling approaches is chosen here. A model falling with the range of this class is advocated in Kintzel and Mosler (2010, 2011), see also Khan et al. (2010), Kintzel et al. (2010) and Khan et al. (2012). In line with Lemaitre (1992) and Lemaitre and Desmorat (2005), this model captures the interplay between elastoplastic deformation and material damage. However, and in sharp contrast to Lemaitre (1992) and Lemaitre and Desmorat (2005), it accounts for ductile damage as well as for quasi-brittle damage. Accordingly, it can model a broad range of different types of fatigue (low cycle fatigue is associated with ductile damage, while high cycle fatigue corresponds at the macro-scale to quasi-brittle damage).

The fatigue model presented in Kintzel and Mosler (2010, 2011) is derived in an isothermal and geometrically linearized setting. The extensions necessary for the more general case are precisely the focus of this paper. However, since the focus of this paper is LCF, only the ductile damage evolution is considered here.

Thermomechanically coupled fatigue models have already been proposed before. In the case of a geometrically linearized setting, such coupled models can be found in Egner (2012), Benallal and Bigoni (2004) and da Costa-Mattos and Pacheco (2009), while the works (Srikanth and Zabaras, 1999; Vaz et al., 2011; Lestriez et al., 2004) are associated with a geometrically exact framework. However, none of the cited models is based on a variational structure, whereas the novel model presented in this paper is. To be more precise, and in line with Ortiz and Stainier (1999), Carstensen et al. (2002), Canadija and Mosler (2011), Mosler and Bruhns (2009), Bleier and Mosler (2012), Yang et al. (2006), Stainier and Ortiz (2010) and Bartels et al. (2015), all unknowns follow jointly and naturally from the stationary condition of an incrementally defined energy potential.

In addition to its physical and mathematical elegance, the aforementioned variational framework also provides some guidance for thermomechanical coupling. The probably most frequently applied approach for this coupling in inelastic materials goes back to the pioneering work by Taylor and Quinney (1934). In this paper it is reported that a certain fraction of the plastic work is transformed to heat. This fraction is nowadays known as Taylor–Quinney-factor. This factor is often assumed as constant, cf. Lehmann and Blix (1985), Simo and Miehe (1992), Canadija and Brnic (2004), Srikanth and Zabaras (1999) and Wriggers et al. (1992). However, experiments such as those performed in Rosakis et al. (2000) and Hodowany et al. (2000) suggests a non-constant factor. Furthermore, models based on the Taylor–Quinney-factor are known as thermodynamically inconsistent, i.e., fulfillment of the second law of thermodynamics is usually not guaranteed. These two fundamental problems are analyzed in detail in Hakansson et al. (2005) and Ristinmaa et al. (2007) and alternative – thermodynamically consistent – definitions of the factor are proposed. More recently, the thermomechanically coupled problem was reanalyzed within the framework of *variational constitutive updates*, cf. Yang et al. (2006), Stainier and Ortiz (2010), Canadija and Mosler (2011) and Bartels et al. (2015). According to Bartels et al. (2015) it is shown that a thermodynamically consistent and realistic temperature increase due to plastic deformations can be predicted by correctly defining the decomposition of the total energy into stored and dissipative parts. Alternatively, and as presented in this paper, the pre-loading history of the considered specimen can be accounted for by non-zero initial internal variables.

The paper is organized as follows. First, the fundamentals of thermomechanically coupled elastoplasticity are summarized in Section 2. Subsequently, the extensions of the model proposed in towards a thermomechanically coupled and geometrically exact description are addressed in Section 3. Finally and in line with Ortiz and Stainier (1999), Carstensen et al. (2002), Canadija and Mosler (2011), Mosler and Bruhns (2009), Bleier and Mosler (2012), Yang et al. (2006), Stainier and Ortiz (2010) and Bartels et al. (2015), the resulting model is re-written into a variationally consistent format. The predictive capabilities of the constitutive model are investigated in Section 5. It is shown that by accounting for the pre-loading history of the considered specimen by means of non-zero initial internal variables, the numerically predicted response agrees excellently with the corresponding experimental data.

## 2. Fundamentals of thermomechanically coupled elastoplasticity

For the sake of completeness, a concise summary of the fundamentals of thermomechanically coupled elastoplasticity theory is given here. Further details can be found elsewhere, e.g., in the comprehensive overviews by Lubliner (1997) and Simo (1998).

### 2.1. Kinematics

In line with classic continuum mechanics, the deformation mapping $\boldsymbol{\varphi} : \mathcal{B}_0 \ni \boldsymbol{X} \mapsto \boldsymbol{x} \in \mathcal{B}_t$ is introduced, where $\mathcal{B}_0 \subset \mathbb{R}^3$ is the undeformed reference configuration and $\mathcal{B}_t \subset \mathbb{R}^3$ is the deformed configuration at time $t$. Based on the deformation map, the deformation gradient

$$\boldsymbol{F} := \mathrm{GRAD}\boldsymbol{\varphi} := \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}, \quad J := \det \boldsymbol{F} > 0 \tag{1}$$

is defined in standard manner as a local measure of deformation. Within the elastoplasticity theory at finite strains, the deformation gradient is often multiplicatively decomposed according to Lee (1969), i.e.,

$$\boldsymbol{F} = \boldsymbol{F}^{\mathrm{e}} \cdot \boldsymbol{F}^{\mathrm{p}}, \quad \det \boldsymbol{F}^{\mathrm{e}} > 0,\ \det \boldsymbol{F}^{\mathrm{p}} > 0. \tag{2}$$

This decomposition is also applied here. For homogeneous deformations, $\boldsymbol{F}^{\mathrm{p}}$ is the deformation gradient associated with the fully unloaded mechanical system, whereas $\boldsymbol{F}^{\mathrm{e}}$ is related to the stress response.

By inserting decomposition $(2)_1$ into the spatial velocity gradient

$$\boldsymbol{l} := \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} \tag{3}$$

the two additional velocity gradients

$$\boldsymbol{L}^{\mathrm{p}} := \dot{\boldsymbol{F}}^{\mathrm{p}} \cdot [\boldsymbol{F}^{\mathrm{p}}]^{-1}, \qquad \boldsymbol{l}^{\mathrm{e}} := \dot{\boldsymbol{F}}^{\mathrm{e}} \cdot [\boldsymbol{F}^{\mathrm{e}}]^{-1} \tag{4}$$

are introduced. Here, the superposed dot represents the material time derivative. Accordingly, $\boldsymbol{L}^{\mathrm{p}}$ is a tensor belonging to the intermediate configuration. This tensor measures the rate of plastic deformations. By way of contrast, tensor $\boldsymbol{l}^{\mathrm{e}}$ is associated with the deformed configuration and measures the rate of elastic deformations.

### 2.2. Balance laws

Next, the (standard) balance laws are briefly introduced. First, balance of linear momentum is considered. In material form (reference configuration) and neglecting inertia effects, it reads

$$\mathrm{DIV}\boldsymbol{P} + \rho_0 \boldsymbol{B}_0 = \boldsymbol{0}. \tag{5}$$