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Non-reciprocity in nonlinear elastodynamics

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ABSTRACT

Reciprocity is a fundamental property of linear time-invariant (LTI) acoustic waveguides governed by self-adjoint operators with symmetric Green's functions. The break of reciprocity in LTI elastodynamics is only possible through the break of time reversal symmetry on the micro-level, and this can be achieved by imposing external biases, adding nonlinearities or allowing for time-varying system properties. We present a Volterra-series based asymptotic analysis for studying spatial non-reciprocity in a class of one-dimensional (1D), time-invariant elastic systems with weak stiffness nonlinearities. We show that nonlinearity is neither necessary nor sufficient for breaking reciprocity in this class of systems; rather, it depends on the boundary conditions, the symmetries of the governing linear and nonlinear operators, and the choice of the spatial points where the non-reciprocity criterion is tested. Extension of the analysis to higher dimensions and time-varying systems is straightforward from a mathematical point of view (but not in terms of new non-reciprocal physical phenomena), whereas the connection of non-reciprocity and time irreversibility can be studied as well. Finally, we show that suitably defined non-reciprocity measures enable optimization, and can provide physical understanding of the nonlinear effects in the dynamics, enabling one to establish regimes of "maximum nonlinearity." We highlight the theoretical developments by means of a numerical example.

1. Introduction

Reciprocity is a well-known and fundamental property of linear time-invariant (LTI) acoustic waveguides [1,2], going back to the work of Lord Rayleigh on the theory of sound. In structural acoustics, reciprocity is a fundamental property of LTI elastic systems governed by self-adjoint operators, leading to symmetric Green's functions [3]. For these media (but also, more generally, in linear or linearized mechanics) the Betti-Maxwell reciprocity theorem applies. In a broader context, in LTI (possibly inhomogeneous) waveguides, reciprocity is directly related to time-reversal symmetry through the Onsager-Casimir principle of microscopic reversibility [4–6]. Accordingly, the break of reciprocity in LTI elastodynamics is only possible through the break of time reversal symmetry on the micro-level [7], but not necessarily on the macro-level. For example, in linear absorbing media with linear viscous dissipation, although time reversal symmetry is broken on the macro-level, reciprocity is still preserved since time reversibility holds on the micro-level.

One way to break reciprocity in LTI elastic systems is by introducing an odd-symmetric external bias, for example, a unidirectional static magnetic field, or a unidirectional fluid circulation. An example was given by Fleury et al. [8], where an

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acoustic analog of the Zeeman electromagnetic non-reciprocal phenomenon was applied to a resonant ring acoustic cavity biased by a circulating fluid with a constant direction of rotation. This led to giant non-reciprocity (that is, to complete elimination of reciprocity), and resulted in preferential sound transmission [9]. Tsakmakidis et al. [10] proposed a broadband high-Q optical cavity connected to a semiconductor-dielectric-metal waveguide by breaking reciprocity of light propagation by means of an external unidirectional magnetic field. This changes the widely-held view that any type of system having a given bandwidth can interact with an incident wave only over a restricted time period which is inversely proportional to the bandwidth.

Alternative ways to break the conditions of the Onsager-Casimir principle and induce non-reciprocity is by introducing nonlinearities or time-variant properties. An example of the latter approach was given by Fleury et al. [11] with coupled acoustic cavities whose volumes were harmonically modulated, imparting an effective angular momentum bias. Another approach employed nonlinear active metamaterials [12]. Regarding nonlinear acoustic non-reciprocity [13], Liang et al. [14,15] proposed an acoustic diode by coupling a superlattice with a nonlinear medium, whereas Boechler et al. [16] and Li et al. [17] used granular media for nonlinear acoustic switching, rectification and logic. Finally, Zhang et al. [18] studied nonlinear non-reciprocity phenomena in a geometrically nonlinear lattice in the plane which, in the limit of low-energy, behaved as a "sonic vacuum" (i.e., it had zero linearized speed of sound). Non-reciprocal dynamics was in the form of targeted energy transfers from large to small length scales, whereas non-reciprocal acoustics involved irreversible nonlinear wave interactions between predominantly transverse propagating localized pulses and axial (sonic) traveling waves.

To date there is no general theoretical framework for nonlinear non-reciprocity in nonlinear elastic systems. Accordingly, the main goal of the present work is to develop a theoretical framework for non-reciprocity in nonlinear elastodynamics that is applicable to a broad class of nonlinear time-invariant systems. Focusing on elastic systems with smooth stiffness nonlinearities, we develop conditions for non-reciprocal response based on multi-harmonic Volterra series expansions and asymptotic analysis. Apart from formulating necessary and sufficient conditions for nonlinear acoustic and dynamic non-reciprocity, we aim to develop quantitative non-reciprocity measures based on which optimization procedures can be developed, and, in the process, to reveal the passively self-tuning nature of non-reciprocal nonlinear elastic systems. Then, we provide an example that highlights the theoretical results and conclude by providing a synopsis of the main findings of this work.

2. Asymptotic analysis based on Volterra series (VS) expansions

Our task to develop a general theoretical framework for acoustic non-reciprocity in broad classes of discrete and continuous systems with localized or spatially extended nonlinearities is based on multi-harmonic Volterra series (VS) expansions. The use of VS in nonlinear systems is not new [19–22], and it enables the construction of higher-order transfer functions and spectra [23,24] to generalize the linear concept of single-frequency transfer function. In particular, a VS expansion represents the response of a nonlinear dynamical system in a functional series of multi-convolution integrals which are considered as higher-order impulse response functions or Volterra kernels. The higher-order spectra are defined as the multi-dimensional Fourier transforms of the Volterra kernels. Convergence criteria for VS were studied for weakly nonlinear systems [25–28], and, in general, are met provided that the dynamical systems under consideration possess sufficiently smooth nonlinearities. In the following analysis we will assume that the developed VS representations are convergent. This clearly holds for the systems with polynomial-type stiffness nonlinearities (e.g., cubic stiffness terms) considered herein, whereas non-smooth nonlinearities such as clearances, vibro-impacts or friction are omitted from consideration in this work.

We consider the general one-dimensional (1D), undamped elastic waveguide governed by the following nonlinear partial differential equation,

$$u_{tt}(x,t) = \mathcal{L}[u(x,t)] + \mathcal{N}[u(x,t)] + \vartheta f(x,t), \quad (x,t) \in \Omega \times \mathbb{R}^+ \tag{1}$$

defined over the spatial domain Ω, and subject to the homogeneous LTI boundary conditions $\mathcal{B}[u] = 0$ on $\partial\Omega$. In (1), the quantity $u(x, t)$ is a scalar response (displacement) field with x and t being the spatial and temporal variables, respectively, $\mathcal{L}[\cdot]$ a self-adjoint LTI operator, $\mathcal{N}[\cdot]$ an essentially nonlinear (i.e., non-linearizable) time-invariant operator, $f(x, t)$ the external forcing function defined in Ω for $t \geq 0$, and ϑ is a small parameter ($|\vartheta| \ll 1$) which will be used in the following asymptotic analysis. Linear viscous damping can be included in this formulation but we omit it in order to more clearly study the connection between stiffness nonlinearity and non-reciprocity; moreover, extension of the analysis to higher dimensions is straightforward from a mathematical point of view, but we refer to our comments in §4 for the new non-reciprocal physical wave phenomena that are expected in higher dimensions. *We aim to study conditions for acoustic non-reciprocity in system (1) by applying a punctual excitation at a given position and studying the symmetry (reciprocity) properties of the response with respect to the points of excitation and measurement.* To this end, we assume that the nonlinear operator in Eq. (1) is sufficiently smooth that it may be Taylor-expanded in the form $\mathcal{N}[u] = a_2u^2 + a_3u^3 + \mathcal{O}(u^4)$, $a_i \in \mathbb{R}$.

To proceed, we consider an impulsive excitation applied at position $x = x_0$ and express the forcing function as $f(x, t; x_0) = \delta(x - x_0)\phi(t)$, with the system initially at rest. Assuming small responses (so that we confine our attention to the weakly nonlinear regime), expressing the resulting response in power series in terms of the small parameter, $u = \sum_{m=1}^{N} \vartheta^m u_m$, substituting into Eq. (1), and matching terms of same powers of ϑ, we obtain a hierarchy of sub-problems at different orders of approximation.