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The dynamics of large-scale arrays of coupled resonators

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ABSTRACT

This work describes an analytical framework suitable for the analysis of large-scale arrays of coupled resonators, including those which feature amplitude and phase dynamics, inherent element-level parameter variation, nonlinearity, and/or noise. In particular, this analysis allows for the consideration of coupled systems in which the number of individual resonators is large, extending as far as the continuum limit corresponding to an infinite number of resonators. Moreover, this framework permits analytical predictions for the amplitude and phase dynamics of such systems. The utility of this analytical methodology is explored through the analysis of a system of N non-identical resonators with global coupling, including both reactive and dissipative components, physically motivated by an electromagnetically-transduced microresonator array. In addition to the amplitude and phase dynamics, the behavior of the system as the number of resonators varies is investigated and the convergence of the discrete system to the infinite-N limit is characterized.

1. Introduction

Coupled dynamical systems have garnered increasing interest over the past three decades due to their ability to describe the complex behaviors attendant to fields such as mathematical biology (see, for example [1,2]) and their ability to describe collective and emergent behavior in physics and engineering contexts. For example, coupled ensembles of oscillators or resonators have been used in the modeling, analysis, design, and characterization of fiber lasers, oil pipelines, bladed disk assemblies, and antennas, amongst other pertinent systems (see, for example, [3–6]). One area of engineering research that has recently advanced the understanding of coupled dynamical systems is that related to micro- and nanoelectromechanical systems, or so-called MEMS and NEMS (see, for example, [7–21]). In this area of application there exist natural economies of scale, which allow for very large degree-of-freedom ensembles, accompanied by noise and parametric uncertainty. The emergent dynamics of such systems are often avoided, yet with proper system design, the global dynamics can be exploited to provide system performance that cannot be realized with individual components. Accordingly, these systems offer a certain appeal for the traditional analyst, as well as the practitioner who values the utility of collective and emergent behaviors in applications as broad as mass sensing, signal processing, pattern generation, and even neural computing.

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Despite the rapid growth in interest related to coupled dynamical systems, analytical advancements in this area have been comparatively slow. Today, most analytical methods are based on mathematical reductions to phase-only models or leverage system symmetries to predict and explain observed dynamical behaviors. Unfortunately, these methods are ill-suited for many practical problems, which feature *both amplitude and phase* dynamics, inherent element-level parameter variation (mistuning), nonlinearity, and/or noise.

This work seeks to help fill this apparent analytical gap through the presentation of a systematic analytical approach which is amenable to the aforementioned problems and is adapted from prior work by [22]. This analytical framework utilizes: (i) a discrete-to-continuous system transformation, followed by (ii) the application of a perturbation method on the continuum representation and (iii) the use of an iterative numerical method to solve the resulting integral equations, and, finally, (iv) interpretation of the results acquired from the continuous analog or, following an additional transformation, the discrete representation.

The aforementioned framework is explored through an investigation of the dynamics of a specific system of interest: a globally coupled microresonator array with reactive and/or dissipative coupling, together with element-level dynamics that contain weak stiffness nonlinearities. To this end, this work presents not only a new analytical framework suitable for the analysis of large degree-of-freedom, coupled dynamical systems, but also provides an analytical pathway suitable for the design and development of coupled micro- and nanoresonator arrays, facilitating their use in very-large-system-integration (VLSI)-like contexts.

2. A discrete system of coupled resonators

Equations of motion

The coupled dynamical system of interest herein is the electromagnetically-transduced microresonator array that was explored, both analytically and experimentally, by the authors in prior work ([23,12]). This microsystem is actuated by Lorentz forces resulting from interactions between current-carrying conducting metal loops deposited on the resonator's surface and an external magnetic field ([24,25]). Likewise, the system exploits the back electromotive force (EMF) resulting from the conductor passing through the magnetic field for sensing. In this system, global dissipative coupling arises naturally due to the current that identically flows through each individual resonator and the resulting electromagnetic interactions described previously. For this work, this coupling is augmented by a global reactive coupling component for the sake of generality. These coupling terms are in contrast to nearest neighbor coupling that might arise from, for example, localized mechanical interactions ([6]). Accordingly, the differential equations of motion that govern the coupled system's dynamics, assuming finite but not large displacements (i.e. tip deflections approximately one order of magnitude smaller than the beam length), are given by

$$m_i\ddot{z}_i + c_i\dot{z}_i + k_i z_i + \gamma_i z_i^3 - \frac{\alpha}{N}\sum_{j=1}^{N}\dot{z}_j - \frac{\beta}{N}\sum_{j=1}^{N} z_j = f_i(t). \tag{1}$$

Here, each individual resonator in the N-element array is characterized by the index i and has an associated displacement z_i. Likewise, m_i, c_i, and k_i represent the mass, damping, and stiffness of each resonator, while γ_i parameterizes the strength of the nonlinearity. The parameters α and β characterize the overall strength of the dissipative and reactive global respectively. Finally, each resonator is subject to an external time-dependent excitation represented by $f_i(t)$.

3. Continuum model

3.1. Formulation

The discrete system of coupled resonators can be recast in a continuum formulation, allowing for a compact description of the resulting dynamics, similar to the system of coupled oscillators considered by [22]. In particular, the global dissipative and reactive coupling terms can be expressed as integrals over the population of resonators. As an example, the dissipative coupling term in Eq. (1) can be written as

$$\frac{\alpha}{N}\sum_{j=1}^{N}\dot{z}_j = \frac{\alpha}{N}\int_{-\infty}^{\infty}\left(\dot{z}(t;\,n)\sum_{j=1}^{N}\delta(n-s_j)\right)\mathrm{d}n, \quad \text{with} \quad \dot{z}_j \equiv \dot{z}(t;\,s_j), \tag{2}$$

where $\delta(x)$ is a Dirac delta function and the distribution parameter s_j identifies the j^{th} resonator. As a result, the discrete system of coupled resonators can be written as a single integro-differential equation of the form

$$m_s\ddot{z}(t;\,s) + \epsilon c_s\dot{z}(t;\,s) + k_s z(t;\,s) + \epsilon\gamma_s z^3(t;\,s) \quad - \epsilon\alpha\int_{-\infty}^{\infty}\dot{z}(t;\,n)\rho_N(n)\mathrm{d}n - \epsilon\beta\int_{-\infty}^{\infty} z(t;\,n)\rho_N(n)\mathrm{d}n = \epsilon f_s(t), \tag{3}$$