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Calculation of force distribution for a periodically supported beam subjected to moving loads

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ABSTRACT

In this study, a novel model for a periodically supported beam subjected to moving loads was developed using a periodicity condition on reaction forces. This condition, together with Fourier transforms and Dirac combs, forms a relation between the beam displacement and support reaction forces. This relation explains the force distribution at the supports, and holds for any type of support and foundation behaviors. Based on this relation, a system equivalence for a periodically supported beam is presented in this paper. An application to non-ballasted viscoelastic supports is presented as an example and the results clearly match the existing model. Next, an approximation of real-time responses was developed for the moving loads as periodic series. The comparison shows that this approximation can be used for a limited number of loads if the distances between loads are sufficiently large. The system equivalence for a periodically supported beam is efficient for supports with linear behavior, and could be extended to other behaviors.

1. Introduction

Support systems for rails have been developed throughout the history of the railway industry. This type of system was initially developed with simple wood blocks, and now designed using different components and materials, and can be used without a ballast layer (the so-called non-ballasted railway). There is no analytical model for such a system; however, a model of a ballasted track with discrete supports is probably applicable. This is also applied to models of infinite periodically supported beams through various techniques [1–9]. Based on the wave propagation on periodic structures and Fourier series techniques, Mead [1,2] developed a model with elastic supports and harmonic loads, while Sheng et al. [6,7] developed one with loads from wheel–rail interactions. A periodicity condition was used by Metrikine et al. [3,4] and Belotserkovskiy [5] to solve the system with moving concentrated forces. Nordborg [8,9] applied the Fourier transform method and Floquet's theorem to obtain Green's function in his model. This Green's function formulation is also used by Foda et al. [10] to calculate the response of a beam structure subjected to a moving mass.

In all these studies, the response of the beam and its supports are investigated in a complete dynamic system. To analyze the interaction between the support system and the foundation, Metrikine et al. [3,4] showed that an elastic half-space can be replaced by an equivalent stiffness. This approach suggests a new viewpoint on the interaction of the beam with its supports when separating these two components. In fact, the beam redistributes the moving forces to its supports. There is

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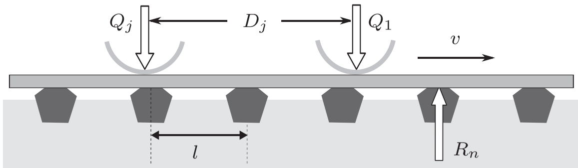

Fig. 1. Periodically supported beam subjected to moving loads.

no existing model that deals with the mechanism of this redistribution. With this aim, a model of periodically supported beams is developed by using another version of the periodicity condition for the steady state included in [3–5]. Furthermore, this condition has always been presented as a boundary condition for beam displacement. In this study, this condition is introduced by considering the periodicity of the support reaction forces. Next, by using Fourier transforms and the Dirac combs, the periodicity condition shows a general relation between the reaction forces and beam displacements. This relation holds true for periodically supported beams with any type of support behavior (linear or nonlinear). Based on this property, a system equivalence with a stiffness and preload for a periodically supported beam is introduced, and presents the force redistribution from the beam to its supports. This new equivalence is independent of the constitutive law of the supports; thus, we can compute the response of the beam and its supports separately.

Next, an application of the system equivalence to non-ballasted supports with viscoelastic behavior is presented as an example, and the results match with those given by Belotserkovskiy [5]. In addition, an approximation of real time response is developed for a periodic series of moving loads. A comparison shows that the approximation can be used when the distance between loads is sufficiently large. This model gives a general viewpoint on the interaction between the beam and its support, even if the support behavior is unknown.

2. System equivalence of a periodically supported beam

Consider a periodically supported beam, with the same constitutive law for all its supports periodically separated by a length l, as shown in Fig. 1. The beam is subjected to moving forces Q_j ($1 \le j \le K$, K is the number of moving forces) characterized by the distance D_j to the first moving force.

Let $R_n(t)$ be the reaction force of a support at the coordinate $x = nl$ (with $n \in \mathbb{Z}$). By considering that these reaction forces are concentrated, we can locate them by utilizing Dirac functions. Therefore, the total force applied on the beam is given by

$$F(x,t) = \sum_{n=-\infty}^{\infty} R_n(t)\delta(x - nl) - \sum_{j=1}^{K} Q_j \delta(x + D_j - vt) \tag{1}$$

When using an Euler–Bernoulli homogeneous beam, the vertical displacement $w_r(x,t)$ of the beam under the total force $F(x,t)$ is solved by the following dynamic equation:

$$EI\frac{\partial^4 w_r(x,t)}{\partial x^4} + \rho S\frac{\partial^2 w_r(x,t)}{\partial t^2} - F(x,t) = 0 \tag{2}$$

where ρ is the density, E is Young's modulus, and S and I are the cross-section and the longitudinal moment of inertia of the beam, respectively.

Eqs. (1) and (2), with initial conditions, establish a relation between the beam displacement $w_r(x,t)$ and reaction forces $R_n(t)$. This relation cannot be calculated analytically because of the infinite number of unknowns. However, we could determine the periodic responses of this linear differential equation if a periodicity condition on the reaction forces is satisfied when the system is stationary (see Floquet's theorem [11]). In the steady state, all supports play the same role and their responses are supposed to be equivalent and unchanged in the reference system of the moving forces. In particular, the reaction forces of all supports are described using the same function but with a delay equal to the time for a moving load to move from one support to another. In other words, the reaction force repeats when a moving force passes from one support to another. That is, $R_n(t) = R(t - \frac{nl}{v})$ and $\hat{R}_n(\omega) = \hat{R}(\omega)e^{-i\omega\frac{nl}{v}}$, where $R(t)$ is the reaction force of the support at $x=0$, and $\hat{R}(\omega)$ its Fourier transform.

The total force (1) in steady-state becomes

$$F(x,t) = \sum_{n=-\infty}^{\infty} R\left(t - \frac{x}{v}\right)\delta(x - nl) - \sum_{j=1}^{K} Q_j \delta(x + D_j - vt) \tag{3}$$

By substituting the last expression into Eq. (2), we obtain a dynamic equation of the beam in steady-state:

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