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Acoustic equations for a gas stream in rigid-body rotation

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A B S T R A C T

The classical topic of wave propagation in a rotating gas is revisited by deducing scalar wave equations for propagation of acoustic and rotational waves through a plug flow of gas in rigid-body rotation with arbitrary intensities of the radial stratification. In the light of these novel equations, wave propagation is analyzed in two different base gas states: isothermal and homentropic. In both cases, previous findings are recovered that assess the validity of the equations and new results are established. In the non-homentropic but isothermal case, the set of governing equations is reduced to two coupled scalar wave equations with space dependent coefficients for the disturbances of density and pressure. Travelling wave solutions with variable amplitude have been obtained in the limit of weak stratification both for inertial waves as for acoustic waves which, in general, propagate on different frequency bands that overlap in the small wavenumber region. Furthermore, the entropy stratification in the base state is stable and compels the propagation of internal waves, leading to hybrid acoustic-inertial-vortical modes. In the homentropic case, the adiabatic relation between pressure and density disturbances allows to reduce further the governing equations to a single fourth-order scalar wave equation. In this case, the sound propagation velocity depends on the distance to the rotation axis and solutions are found by multiple-scale analyses in the form of waves with slowly varying amplitude and wavenumber. The corresponding eikonal equation shows that acoustic rays are refracted towards the rotation axis, propagating and spinning along and around it. In that way, the swirling gas behaves as an axial waveguide trapping inside any acoustic ray propagating in the vortex with large enough azimuthal and/or vertical wavenumber component.

1. Introduction

There is a vast amount of publications devoted to the classical subject of linear waves in incompressible rotating fluids. Being a linear problem, most of this literature pertains today to text books (see, among them [1–6]). For an incompressible fluid in rigid-body rotation, wave propagation can be described in a co-rotating frame, with respect to which the fluid is at rest. The field of forces induced by the centrifugal acceleration is customarily added to the pressure gradient in the momentum equation leading to an effective hydrostatic *pressure* term. This is also a common practice when dealing with geophysical flows in the atmosphere where gravity is much larger than the centrifugal acceleration and the stratification due to gravity is accounted for by assuming the air density is exponentially decreasing with height. Since the seminal work

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by Sobolev [7], many efforts have been devoted to this subject due to its implications for wave propagation in the atmosphere (see, for instance, [8]). The same idea has been extended to consider also the interaction of waves with atmospheric flows resulting in more complex vertical distributions of density, like the case of a Rankine-type vortex studied by Moulin & Flór [9].

A consequence of the formation of density gradients is the well-known occurrence of acoustic ray refraction, a feature appearing in many natural scenarios. To mention just a few, acoustic wave propagation in stars where, due to the (isotropic) stratification by gravity, the sound speed increases with depth so that acoustic waves emitted downwards into the interior are refracted back up to the star surface [10], as also happens to seismic P-waves in the Earth mantle [11]. Another more striking example, that will be seen more related to the case analyzed here, is the propagation of acoustic waves in the ocean along the so-called SOFAR channel. At some depth, a minimum value of the sound velocity appears due to its combined dependence on pressure and temperature. This layer behaves like a planar waveguide as the acoustic rays follow horizontal zig-zag paths by bending always towards it ([4], p. 251).

When the stratification effects are driven by the centrifugal acceleration, the density gradients resulting from the gas radial compression break the similarity with a simple pressure gradient by introducing space dependent coefficients in the governing differential equations. In general, a radially varying sound velocity ensues from this density gradient with the minimum value at the rotation axis such that any propagating sound wave will bend towards it. Thus, any coherent vortical structure is expected to behave as an axisymmetric acoustic waveguide, at least up to some degree depending on size as well as on the radial distribution and intensity of the rotational velocity. In spite of the simplicity of the swirling flow considered in this work, the essential fact of axisymmetric stratification by swirl is present in all natural vortical structures and the qualitative aspects of the analysis exposed below are expected to hold and be helpful to understand some of the observed phenomenology [12]. These features could also be of relevance for the problem of sound dispersion by a vortical structure [9] where the manifestation of the waveguide effect would have interesting implications on the sound scattering and radiation by a vortex in the limit of small wavelengths compared to the vortex size, opposite to the range of aeroacoustic sound generation [13].

Besides, the subject of waves propagating in a swirling ducted gas has yielded an abundant specialized literature mainly because of its relevance to technical applications (turbomachinery, gyroscopes, centrifuges…). Some of these works have been concerned with boundary conditions problems arising from finite-size effects, which may lead to instability and vortex break-down processes [14,15] or to resonance conditions heading to Sturm-Liouville-type problems. Morton & Shaughnessy [16] were among the first to consider an isothermal gas in rigid-body rotation enclosed in a cylinder and perform a mode decomposition analysis that led to an equation for the radial part of the wave which associated eigenvalue problem was solved numerically. For homentropic flows, similar analyses were pioneered by Kerrebrock [17] to study acoustic instabilities in axial flow machinery, a topic that has been subsequently the subject of a large number of works (for instance, see among them, [18–21] and for a recent review see Peake & Parry [22] and the references cited therein).

Thus the subject covers a large variety of topics operating over a wide range of scales from rotary devices to geophysical or astrophysical phenomena. Among the large number of works related to this subject, little efforts have been devoted to the establishment of simpler scalar wave equations which could provide an easier way to describe the propagation of acoustic and inertial disturbances in such inhomogeneous media. Among these, Erofeev and Soldatov [23] deduced a third-order partial differential equation to describe the waves propagating normally to the rotation axis and, more recently, Posson and Peake [24] have derived a sixth-order partial differential equation for the fluctuating pressure field in an annular duct with body forces.

Following this line, the present work focuses on the quest for scalar acoustic equations and on the analysis of travelling wave solutions in a plug flow of swirling gas with constant rotational velocity. To simplify the analysis, other issues like the finite-size effects cited above which are not essential to the problem of propagation will be left aside. Our aim is to extend the study by Marin-Antuña, Hall & Saad [25] who shown that under some assumptions a single fourth order partial differential equation governs the propagation of linear waves in a rotating gas and considered the dispersion relation of planar acoustic waves. The same type of analysis will be pursued here but including the driving effects of the centrifugal stratification with analytical methodologies [26]. New insights on the problem will be obtained by analyzing some simple but relevant cases when the number of governing equations can be reduced to one or two wave equations, depending on the base gas state.

The organization of the paper is as follows. In the next section, the set of five differential governing equations (the three components of momentum, continuity, and energy) are solved for an ideal gas in a steady spiraling flow (swirl plus axial advection) under two particular thermodynamic constraints: isothermal and homentropic. In Section 3, the governing equations are linearized on the disturbances and written in nondimensional form in terms of the normalized variables and the dimensionless controlling groups. The system of equations for the disturbances is reduced to either two or one scalar wave equations in Sections 4 and 6, respectively, depending on the imposed base condition (isothermal or homentropic). The details of the algebraic manipulations are in the Appendix. Solutions of these equations are obtained in both cases in the form of dispersive travelling waves in the limit of *slow* rotation velocities. The different branches of the dispersion relation, corresponding to inertial and acoustic waves, are obtained in the respective limits whereas, in the geometrical acoustic limit of the homentropic case, the ray trajectories are analyzed by multiple-scale techniques. The results for both cases, isothermal and homentropic, are discussed separately in Sections 5 and 7, respectively. Finally, the main conclusions and perspectives are presented in Section 8.