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On the properties of well-graded partially union-closed families

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HIGHLIGHTS

- Learning space axioms are generalized to partially union-closed families.
- The concept of an upper intersection-closed family is introduced.
- Various other properties of partially union-closed families are studied.

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ABSTRACT

In this paper we will study several properties of well-graded union-closed families that do not contain the empty set. Such union-closed families without the empty set are said to be partially union-closed. We will extend several results for well-graded union-closed families to the partially union-closed case, and we will also extend the concept of being intersection-closed to families without the empty set.

1. Introduction

Families of sets that are ∪-closed are of interest for both their theoretical properties and their use in practical applications. In particular, knowledge spaces are ∪-closed families of sets that have found many successful uses in the assessment of knowledge (Doignon & Falmagne, 1985; Falmagne, Albert, Doble, Eppstein, & Hu, 2013; Falmagne & Doignon, 2011).

Definition 1.1. A *knowledge structure* is a pair $(Q, \mathcal{K})$ in which Q is a nonempty set, and $\mathcal{K}$ is a family of subsets of Q, containing at least Q and the empty set $\varnothing$. The set Q is called the *domain* of the knowledge structure. Its elements are referred to as *questions* or *items* and the subsets in the family $\mathcal{K}$ are labeled *(knowledge) states*. Since $\cup\mathcal{K} = Q$, we shall sometimes simply say that $\mathcal{K}$ is the knowledge structure when reference to the underlying domain is not necessary. If a knowledge structure $\mathcal{K}$ is closed under union, we say that $\mathcal{K}$ is a *knowledge space*.

A useful concept associated with ∪-closed families is well-gradedness, which we will define as in Doignon and Falmagne (1997).

Definition 1.2. Let Δ denote the standard symmetric difference between sets. Then, a family of sets $\mathcal{F}$ is *well-graded* if for any $A, B \in \mathcal{F}$ with $|A\Delta B| = n$, there exists a finite sequence of sets $A = K_0, K_1, \ldots, K_n = B$ in $\mathcal{F}$ such that $|K_{i-1}\Delta K_i| = 1, i = 1, \ldots, n$. The sequence of sets $A = K_0, K_1, \ldots, K_n = B$ satisfying these conditions is called a *tight path* between A and B.

If a knowledge space is well-graded, Cosyn & Uzun (2009) showed that we have a learning space, a special type of knowledge space whose properties are motivated by pedagogical assumptions. One subtle assumption is that the empty set $\varnothing$ is necessarily part of a learning space. While this may not seem like an important assumption at first glance, in what follows we will see that many of the properties of learning spaces, as well as the techniques used to study these properties, depend heavily on the inclusion of the empty set; thus, extra complications arise when the inclusion of the empty set is not guaranteed.

In this paper we will focus on the properties of well-graded ∪-closed families of sets that do not contain the empty set; such families are said to be well-graded and partially ∪-closed. As mentioned in the previous paragraph, the lack of an empty set presents obstacles that will require different techniques to handle compared to those used for a normal ∪-closed family. Starting with several properties of learning spaces, we will derive analogous results for well-graded partially ∪-closed families of sets. Furthermore, we will also extend the concept of an ordinal space (i.e., a

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discriminative learning space that is also ∩-closed) to families of sets that do not contain the empty set. Then, after proving a result for projections of such families, we will finish by looking at the infinite case. Along the way we will have provided possible solutions to two of the open problems mentioned in Section 18.2 of Falmagne and Doignon (2011).

In addition to the interesting theoretical challenges that result from the lack of an empty set, there are practical reasons for studying such families. As described in Falmagne (electronic preprint) and Chapters 2 and 13 of Falmagne and Doignon (2011), partially ∪-closed families may be encountered when using projections of knowledge spaces. Such projections of knowledge spaces have found applications in assessments of knowledge where, in many cases, it may be unwieldy, or even impossible, to run an assessment over a full knowledge space.

As another example of a practical application, when a ∪-closed family is being used to represent a domain of knowledge, one can make the argument that the empty set is not a realistic state in many situations. In an implementation of knowledge spaces such as the artificial intelligence used in the ALEKS system, having a student in a knowledge space with the empty set as their state would seem to indicate that the student is misplaced; in reality, all students have some level of knowledge, so it is likely that there exists a different knowledge space that would be a better fit for such a student. Under this viewpoint, a student who is placed in a properly designed domain of knowledge should never start in the empty state. The benefit is that the family of sets can then be engineered and built without having to necessarily include the empty set. Thus, by starting from a collection of minimal nonempty sets, the entire family can be built without needing to worry about the empty set or any other sets contained in these minimal states. In essence, the process can be simplified by not having to worry about the "bottom" of the family of sets.

2. Background

Motivated by pedagogical assumptions, Cosyn & Uzun (2009) introduced two axioms that define a learning space (note that, with the exception of Section 6, we will assume throughout this paper that we are dealing with a finite family of sets on a finite domain of items).

Definition 2.1. A knowledge structure $(Q, \mathcal{K})$ is called a learning space if it satisfies the following conditions.

[L1] LEARNING SMOOTHNESS. For any two states K, L such that $K \subset L$, there exists a finite chain of states

$$K = K_0 \subset K_1 \subset \cdots \subset K_p = L$$

such that $|K_i \setminus K_{i-1}| = 1$ for $1 \leq i \leq p$ and so $|L \setminus K| = p$.

[L2] LEARNING CONSISTENCY. If K, L are two states satisfying $K \subset L$ and q is an item such that $K \cup \{q\} \in \mathcal{K}$, then $L \cup \{q\} \in \mathcal{K}$.

Cosyn and Uzun showed that a learning space, characterized by these axioms, is equivalent to a well-graded ∪-closed family.

Theorem 2.2 (*Cosyn and Uzun*)**.** *Let $\mathcal{F}$ be a family of sets containing the empty set. Then, $\mathcal{F}$ is well-graded and ∪-closed if and only if [L1] and [L2] are satisfied. In other words, well-graded ∪-closed families of sets are characterized by axioms [L1] and [L2].*

The example below (copied from Example 2.2.8 in Falmagne & Doignon, 2011) shows that Theorem 2.2 fails to hold when $\mathcal{F}$ does not contain the empty set. In particular, for a family $\mathcal{F}$ without the empty set, [L1] and [L2] do not guarantee that $\mathcal{F}$ is well-graded or ∪-closed.

Example 2.3. The family of sets

$$\mathcal{L} = \{\{a\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$$

satisfies [L1] and [L2]. However, $\mathcal{L}$ is neither ∪-closed nor well-graded.

In Section 3 we will derive a set of axioms that gives a result analogous to Theorem 2.2 when $\mathcal{F}$ does not contain the empty set. To do this, we will make use of the terminology in the following definition from Falmagne & Doignon (2011).

Definition 2.4. A family $\mathcal{F}$ of subsets of a nonempty set Q is a *partial knowledge structure* if it contains the set $Q = \cup\mathcal{F}$. We also call the sets in $\mathcal{F}$ *states*. A partial knowledge structure $\mathcal{F}$ is a *partial learning space* if it satisfies axioms [L1] and [L2]. A family $\mathcal{F}$ is *partially ∪-closed* if for any nonempty subfamily $\mathcal{G}$ of $\mathcal{F}$, we have $\cup\mathcal{G} \in \mathcal{F}$. (Contrary to the ∪-closure condition, partial ∪-closure does not imply that the empty set belongs to the family). A *partial knowledge space* $\mathcal{F}$ is a partial knowledge structure that is partially ∪-closed.

We will also need the following definition from Eppstein, Falmagne, & Uzun (2009).

Definition 2.5. Let $\mathcal{F}$ be a nonempty family of sets. For any $q \in Q = \cup\mathcal{F}$, an *atom at* q is a minimal set of $\mathcal{F}$ containing q (where 'minimal' is with respect to set inclusion). A set $X \in \mathcal{F}$ is called an *atom* if it is an atom at q for some $q \in Q$. We denote by $\sigma(q)$ the collection of all atoms at q, and we call $\sigma : Q \to 2^{2^Q}$ the *surmise function* of $\mathcal{F}$. We say that σ is *discriminative* if whenever $\sigma(q) = \sigma(q')$ for some $q, q' \in Q$, then $q = q'$. In such a case, we also refer to the family $\mathcal{F}$ as *discriminative*.

Throughout this paper we will assume that $Q = \cup\mathcal{F}$ and, with the exception of Section 6, that Q is a finite set.

3. Axioms for well-graded partially ∪ -closed families

Consider the following axioms for a family $\mathcal{F}$ of sets (states) that does not contain the empty set.

Definition 3.1. [L1] (same as in Definition 2.1) For any two states K, L such that $K \subset L$, there exists a finite chain of states

$$K = K_0 \subset K_1 \subset \cdots \subset K_p = L$$

such that $|K_i \setminus K_{i-1}| = 1$ for $1 \leq i \leq p$ and so $|L \setminus K| = p$.
[L2*] Let A be an atom, and let $q_1, \ldots, q_n \in Q$ be all the items in Q at which A is an atom. If L is a state such that $A \setminus \{q_1, \ldots, q_n\} \subseteq L$, then $L \cup \{q_1, \ldots, q_n\} \in \mathcal{F}$.

The next result shows that, in the case of a family of sets containing ∅, [L1] and [L2*] are equivalent to [L1] and [L2].

Theorem 3.2. *Let $\mathcal{F}$ be a family of sets. Then, [L1] and [L2] hold whenever [L1] and [L2*] hold. Furthermore, [L1] and [L2] are equivalent to [L1] and [L2*] when $\mathcal{F}$ contains the empty set.*

Proof. We will first show that [L1] and [L2] are implied by [L1] and [L2*]. Note that for this initial result we will not assume that $\mathcal{F}$ contains the empty set. Let $K, L \in \mathcal{F}$ with $K \subset L$, and let $q \in Q$ be such that $K \cup \{q\} \in \mathcal{F}$. For [L2] to hold, we need to show $L \cup \{q\} \in \mathcal{F}$. Let A be an atom at q such that $A \subseteq K \cup \{q\}$. For $q_1 = q$ and $n \geq 1$, let $\{q_1, \ldots, q_n\} \subseteq A$ be the set composed of all the items at which A is an atom. It follows that $A \setminus \{q_1, \ldots, q_n\} \subseteq K \subset L$, and by [L2*] we have $L \cup \{q\} = L \cup \{q_1, \ldots, q_n\} \in \mathcal{F}$.

We will next assume that $\mathcal{F}$ contains the empty set. Given [L1] and [L2], it is shown in Cosyn and Uzun (2009) that $\mathcal{F}$ is a well-graded ∪-closed family. By Theorem 5.4.1 in Falmagne and Doignon (2011) (which is a generalization of a result from Koppen,