Journal of Mathematical Psychology ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Journal of Mathematical Psychology

journal homepage: www.elsevier.com/locate/jmp

On the assessment of learning in competence based knowledge space theory

Luca Stefanutti *, Debora de Chiusole

University of Padua, Italy

HIGHLIGHTS

- In CbKST the student's skills are inferred from her responses to a set of items.
- There is no one-to-one correspondence between competence and performance states.
- There could be no way for establishing whether learning has occurred or not.
- We establish conditions for resolving this impasse.

ARTICLE INFO

Article history:
Received 4 March 2016
Received in revised form 10 August 2017
Available online xxxx

Keywords:
Competence based-knowledge space theory
Knowledge assessment
Learning assessment
Well-graded competence space
Outer fringe
Exclusive skill function

ABSTRACT

The objective of an assessment in competence based-knowledge space theory (Cb-KST) is to infer the skills of an individual from her responses to a subset of problems. A major issue in this approach is the lack of a one-to-one correspondence between the competence states and performance states. The assessment is possible, but it cannot go beyond an approximation. The problem becomes even more serious if Cb-KST is used for the assessment of learning, since changes caused at the competence level may not be represented by changes at the performance level. The consequence is that there could be no way for establishing whether learning has occurred or not. This impasse can be resolved for the class of conjunctive skill functions by pretending that the competence space induced by the skill function is well-graded. Under this condition an individual can make tangible progresses along the performance structure by learning one skill at a time, until full mastery is eventually reached. If the competence structure is a space satisfying certain compatibility conditions and the skill function satisfies a special property, called *exclusiveness*, then well-gradedness can be assured. A test of exclusiveness of a conjunctive skill function is described and exemplified.

1. Introduction

The theory of knowledge spaces (KST, Doignon & Falmagne, 1985, 1999; Falmagne & Doignon, 2011; Falmagne, Koppen, Villano, Doignon, & Johanessen, 1990) provides a valuable mathematical framework for the development of computerized web-based systems for the assessment of knowledge and learning. The very basic and central notion of the whole theory is that of a *knowledge state*, which is operationally defined as the set K of all those problems an individual is capable of solving in a finite domain of knowledge Q. As a single individual is characterized by a knowledge state, a whole population of individuals is represented by a *knowledge structure*, which is a pair $(Q, \mathcal{K})$ where $\mathcal{K}$ is a collection of knowledge states containing at least Q and the empty set.

A special type of knowledge structure is called *knowledge space*, and plays a central role in KST because of the *closure under union* property which is assumed on its collection of states. A knowledge structure $\mathcal{K}$ is closed under union when, given any subfamily $\mathcal{F} \subseteq \mathcal{K}$, the union $\bigcup \mathcal{F}$ is in $\mathcal{K}$. This assumption inspired the KST's authors since it has a reasonable empirical interpretation: if two students having two different knowledge states are involved in an extensive interaction, it is plausible that, at some point, their knowledge arise from the union of their initial knowledge states. Of course, this situation may not happen, but the knowledge structure should cover this case.

As far as learning is concerned, it is conceivable that a student whose knowledge state is K, after learning some new material will end up to a new knowledge state K' that, if no forgetting occurs, is a strict superset of K. If the collection of all possible knowledge states is correctly represented by $\mathcal{K}$, then learning can be described as a chain $K = K_1 \subset K_2 \subset \cdots \subset K_n = K'$ of knowledge states in $\mathcal{K}$. Behind this description of learning there is the simple idea that,

* Corresponding author.
E-mail address: luca.stefanutti@unipd.it (L. Stefanutti).

http://dx.doi.org/10.1016/j.jmp.2017.08.003
0022-2496/

in moving from the empty set of items to the total mastery Q, a student must traverse a number of states of intermediate mastery.

For a student in knowledge state $K \in \mathcal{K}$, the "smallest possible learning step" would consist of learning exactly one new item, among those in $Q \setminus K$. Under a pedagogical perspective, this "smallest step" plays an important role, since it represents the easiest way a student has for making some tangible progress in learning. The notion of *outer fringe* (Doignon & Falmagne, 1999) formalizes these ideas. For an arbitrary knowledge state $K \in \mathcal{K}$, it is defined as

$$K^{\circ} = \{q \in Q \setminus K : K \cup \{q\} \in \mathcal{K}\}.$$

Informally, the outer fringe of K is regarded as "what the student is ready to learn from her knowledge state". It should be noted that the outer fringe of a knowledge state K could even be empty, meaning that there is no way for a student in knowledge state K to make progresses by learning exactly one new item.

There is a special class of knowledge structures in which every knowledge state has a nonempty outer fringe. They are known as the *well-graded knowledge spaces* or, in the more recent literature on KST (Falmagne, Albert, Doble, Eppstein, & Hu, 2013; Falmagne & Doignon, 2011) *learning spaces*. A learning space is a $\cup$-closed subfamily of 2^Q in which the additional condition holds true that, for every nonempty knowledge state $K \in \mathcal{K}$ there is some item $q \in K$ such that $K \setminus \{q\} \in \mathcal{K}$.

"Being capable of solving a problem", the very basic statement at the core of the definition of a knowledge state, is a very pragmatical one. Early developments of KST did not pay much attention to "how problems are solved by individuals", or to which skills, abilities and knowledge are involved in the solution of the problems belonging to the knowledge state of an individual. Indeed, rather than problems, what people learn are the skills and the knowledge necessary for solving problems. Skills that have been learned from the solution of one problem can then be transferred by individuals to the solution of other problems.

A number of theoretical extensions have been worked out by various authors (Doignon, 1994; Düntsch & Gediga, 1995; Falmagne et al., 1990; Gediga & Düntsch, 2002; Heller, Stefanutti, Anselmi, & Robusto, 2015; Heller, Ünlü, & Albert, 2013; Korossy, 1993, 1997, 1999) to incorporate the cognitive level of the skills into the theory. Basic KST equipped with these extensions is known as *competence based-knowledge space theory* (Cb-KST, Heller, Augustin, Hockemeyer, Stefanutti, & Albert, 2013; Heller, Ünlü et al., 2013) and is divided into two parallel and interdependent levels: the *performance level* and the *competence level*. The knowledge domain Q and the knowledge state $K \subseteq Q$ of an individual are located at the performance level. At the competence level a set S of skills, required for solving the problems in Q, is assumed and the individual knowledge is represented by a subset $C \subseteq S$, named the *competence state*.

The two levels are connected by two mappings, known as the *skill function* and the *problem function* (Düntsch & Gediga, 1995). By assigning skills to problems, the skill function goes from the performance level to the competence level. The problem function instead goes in the opposite direction: given any competence state C, it specifies the subset $K \subseteq Q$ of problems that can be solved by C.

Cb-KST has been used as the reference theory in the development of existing computer systems for skill assessment and learning at the various school grades and for high-level education. To give some examples, we mention the APeLS system (Hockemeyer, Conlan, Wade, & Albert, 2003), for learning in Newtonian mechanics, the iClass system for self-regulated personalized learning (Heller, Augustin et al., 2013), and the two educational games ELEKTRA (http://www.elektra-project.org/) and 80Days (http://www.eightydays.eu/) developed within projects funded by the European Commission. Furthermore, the Knowlab prototype (http://www.knowlab.org/) has been developed at the University of Padua and it is currently applied for the assessment and learning of statistics at the University level. Some probabilistic models together with empirical applications were also developed in Cb-KST (Anselmi, Robusto, & Stefanutti, 2012, 2013; Anselmi, Stefanutti, de Chiusole, & Robusto, 2017; de Chiusole, Anselmi, Stefanutti, & Robusto, 2013; de Chiusole & Stefanutti, 2013; Lukas & Albert, 1993; Robusto, Stefanutti, & Anselmi, 2010; Stefanutti, Anselmi, & Robusto, 2011).

A parallel framework in which cognitive assessment was studied in depth, is the theory of the cognitive diagnostic models (CDM; Bolt, 2007; de la Torre, 2009; DiBello & Stout, 2007; Junker & Sijtsma, 2001; Tatsuoka, 2002, 2009) in which the main interest is at the competence level. There are close connections between CDM and Cb-KST, as recently pointed out by Heller et al. (2015).

The objective of an assessment in Cb-KST is to infer the competence state of an individual from her responses (coded as "correct" or "wrong") to some suitable subset of problems in Q. Essentially, the assessment involves two separate tasks: (1) infer the knowledge state K from the responses to the problems and (2) derive the competence state C from K. In task (1) inference is usually probabilistic, due to the fact that the answer to a problem could be the result of a careless error or a lucky guess (Falmagne & Doignon, 1988a, b). Task (2) instead is deterministic and it is based on the problem function. Since this function is a mapping from competence states to knowledge states, if it is a bijection, its inverse can be applied for inferring the competence state corresponding to the knowledge state assessed in step (1). However, as pointed out by Heller et al. (2015), the problem function might fail to be a bijection and therefore an inverse function could not exist. This fact poses an unescapable problem to knowledge and learning assessment under the framework of Cb-KST: individual assessment cannot go beyond an approximation, in which only a portion of the full set S of skills can be classified as "mastered" or "not mastered", whereas the status of the remaining skills is unknown.

The problem becomes even more stringent when repeated assessments are carried out in different occasions on the same individual with the objective of monitoring learning. This is a typical task, for instance, of a computer-based tutoring system, which constantly switches between assessment sessions and teaching and training sessions. Due to the lack of a one-to-one correspondence between the competence states and the knowledge states, a change in the competence state might not be reflected by a corresponding change in the knowledge state. In the worst situation, changes at the competence level could even produce no change at all at the performance level. This could be a serious problem for any tutoring system developed under the Cb-KST framework.

The theoretical work presented in this article develops upon the notion of an *effective skill*. Once learned by an individual in competence state C, an effective skill produces a corresponding change in her knowledge state. If such a special skill exists and can be pointed out for every possible competence state, then an individual can always make observable progresses by gradually learning one skill at a time, until full mastery is eventually attained. For the class of the conjunctive skill functions it is shown that the existence of (at least) one effective skill per competence state can be assured if the collection of the so-called *minimal competence states* is a well-graded space. A consequence of these requirements is that they assure the union-closure of the competence structure, and the intersection-closure of the knowledge structure. If, at a first look, this situation seems incoherent with the original KST theory, in which the $\cup$-closure of the knowledge structures was recommended, the nice result is that this property moves from the performance level to the competence level, which is the focus in Cb-KST.