Journal of Mathematical Behavior 48 (2017) 1–13

Contents lists available at ScienceDirect

Journal of Mathematical Behavior

journal homepage: www.elsevier.com/locate/jmathb

Students' conceptualisations of multiplication as repeated addition or equal groups in relation to multi-digit and decimal numbers

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ARTICLE INFO

Keywords:
Models for multiplication
Calculations
Arithmetical properties
Connections
Multi-digit numbers
Decimal numbers

ABSTRACT

Multiplicative understanding is essential for mathematics learning and is supported by models for multiplication, such as equal groups and rectangular area, different calculations and arithmetical properties, such as distributivity. We investigated two students' multiplicative understanding through their connections between models for multiplication, calculations and arithmetical properties and how their connections changed during the school years when multiplication is extended to multi-digits and decimal numbers. The case studies were conducted by individual interviews over five semesters. The students did not connect calculations to models for multiplication, but showed a robust conceptualisation of multiplication as repeated addition or equal groups. This supported their utilisation of distributivity to multi-digits, but constrained their utilisation of commutativity and for one student to make sense of decimal multiplication

1. Introduction

The development of multiplicative understanding is a process lasting many years (Vergnaud, 1994). Initially students learn to coordinate composite numbers at several levels of abstraction and to recognise multiplicative situations (Steffe, 1992). This structural level is coincident with the development of calculative procedures in the domain of single-digits, in which initial counting strategies evolve, over repeated addition, to memorised number facts (Fuson, 2003). These single-digit calculations are challenged when multiplication is expanded to multi-digit and decimal numbers (De Corte & Verschaffel, 1996; Verschaffel, Greer, & De Corte, 2007). Multi-digit calculations undertaken by repeated addition are cumbersome (Ambrose, Baek, & Carpenter, 2003) and made problematic by decimal numbers; it is difficult to add repeatedly a number 4.6 times (De Corte & Verschaffel, 1996). Additionally, decimals smaller than 1 challenge the intuitive rule that 'multiplication makes bigger', (De Corte & Verschaffel, 1996; Fischbein, Deir, Nello, & Marino, 1985); prompting a need for models for multiplication that explain such unexpected results (Greer, 1992). Finally, different multiplicative situations are best represented by different models for multiplication (Fuson, 2003; Verschaffel & De Corte, 1997) as well as calculation strategies underpinned by arithmetical properties (Schifter, Monk, Russel, & Bastable, 2008). However, there is little knowledge concerning what connections students draw between models for multiplication, calculations and arithmetical properties at the stage when multiplication is extended from single-digit to multi-digit and decimal numbers. Considering that connections between different types of knowledge and between different aspects of a concept is seen as a sign of deep understanding (Barmby, Harries, Higgins, & Suggate, 2009; Baroody, Feil, & Johnson, 2007; Hiebert & Carpenter, 1992) we identified a need for studies focussing students' connections. In this paper, we report of two case studies in which we investigated not only how students connect models for multiplication, calculation and arithmetical properties but also how these connections can change over time.

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http://dx.doi.org/10.1016/j.jmathb.2017.07.003
Received 16 August 2016; Received in revised form 14 April 2017; Accepted 6 July 2017
0732-3123/

2. Literature review

In the following we examine literature on students' multiplicative understanding at the point where multiplication is extended from single-digit to multi-digit and decimal numbers. In so doing we examine students' understandings of the arithmetical properties, since they underpin multiplicative calculation strategies, as well as those multi-digit calculations that reflect an overgeneralisation of addition, since such errors pose obstacles to the development of multiplicative understanding. We also review literature concerning connections as signs of conceptual understanding. But first we examine four models for multiplication known to play important roles in students' understanding of multiplication.

2.1. Models for multiplication

Different models for multiplication highlight different aspects of multiplication (Barmby et al., 2009). Four models of interest to this study are equal groups, rectangular array, rectangular area, and multiplicative comparison. An example of an equal groups model is four bags of six cookies, while a rectangular array could be four rows of six cookies placed orthogonally on a tray. Rectangular area draws on continuous measures of length transformed to area by multiplication, making area conceptually different from an array of discrete objects (Greer, 1992). Finally, an example of a multiplicative comparison model is that Sofia has four times as many marbles as Martin, who has six, thus there are two sets of marbles: Sofia's twenty-four and Martin's six. Equal groups is commonly the initial model for multiplication in instruction (Greer, 1992; Izsák, 2005) connecting multiplication to the calculation procedure of repeated addition (De Corte & Verschaffel, 1996).

Models for multiplication are either symmetrical or asymmetrical (Greer, 1992). Symmetrical models such as rectangular arrays and area facilitate the learning of commutativity (Schliemann, Araujo, Cassundé, Macedo, & Nicéas, 1998). Equal groups and multiplicative comparisons are asymmetrical; the factors have different roles, which makes commutativity covert (Lo, Grant, & Flowers, 2008; Schliemann et al., 1998). In four bags of six cookies, four (bags) is the multiplier and six (cookies) the multiplicand; it is not evident that six bags of four cookies would be as many (Barmby et al., 2009; Greer, 1992).

Multiplicative comparison and rectangular area, in contrast to equal groups and rectangular arrays, permit the use of continuous variables, which facilitates the conceptualisation of decimal multiplication (De Corte & Verschaffel, 1996; Greer, 1992). The area of a rectangle with lengths of 4.6 and 3.8 units is not difficult to imagine, whereas 4.6 bags of 3.8 marbles is unrealistic. Also, having 4.6 times as much money as someone who has 3.80 Euro makes sense.

Students' typically meet different models when solving word problems and there is a direct link between problem posing and problem solving (Cai, Hwang, Jiang, & Silber, 2015). The type of problems that students, and teacher students, pose can reveal conceptual understandings or misunderstandings (Cai et al., 2015; Prediger, 2008; Tichá & Hošpesová, 2013). For example, when asked to pose a problem to fit a calculation like $9 \cdot 3 = 27$, students typically pose an equal groups problem, demonstrating that it is a common conceptualisation of multiplication (De Corte & Verschaffel, 1996). The one-sided use of equal groups model when posing word problems is typically explained by the dominant exposure to that model in early instruction and everyday life (English, 1998; Verschaffel et al., 2007). Problem posing may thus reveal what models students connect to the operation. In addition, models for multiplication such as equal groups, can support calculation strategies for multi-digit multiplication and intuitive use of distributivity (Carpenter, Levi, Franke, & Koehler, 2005; Ding & Li, 2014).

While equal groups is adequate for integers, it fails with decimals (Greer, 1992), with the consequence that students pose fewer realistic problems for decimals, a situation that is even more pronounced with problems like $0.8 \cdot 0.6$ (De Corte & Verschaffel, 1996). Thus, problem posing to decimal multiplication may reflect problems to conceptualise decimal multiplication.

2.2. Arithmetical properties

The commutative property allows a change of the order of the factors in multiplication and addition; $a \cdot b = b \cdot a$ *and* $a + b = b + a$, which reduces memorised number facts by almost half (Fuson, 2003). Young students can develop an understanding of additive commutativity independent of instruction (Canobi, Reeve, & Pattison, 2002), which is not true for multiplication (Ambrose et al., 2003; Schliemann et al., 1998). When problems have a large multiplier and a small multiplicand, unschooled Brazilian street sellers, in contrast to schooled children, have difficulty exploiting commutativity. For example, with respect to 60 objects at the cost of 4 cruzeiros each, unschooled children typically add 4 sixty times, rather than add 60 four times (Schliemann et al., 1998). Alternatively, English 9- and 10-year old students, who are likely to have received instruction, have a better understanding of commutativity (Squire, Davies, & Bryant, 2004).

The distributive property, $a(b + c) = ab + ac$, underpins mental calculation strategies as well as algebra and is considered hard to learn (Carpenter et al., 2005; Ding & Li, 2014). Strategies where both factors are partitioned are thought to demonstrate a high level of understanding of multi-digit multiplication (Ambrose et al., 2003), and to partition both factors by hundreds, tens and ones is the basis for standard algorithms (Fuson, 2003; Izsák, 2004; Lampert, 1986). Despite evidence that students find distributivity troublesome (Squire et al., 2004), young students can understand and employ distributivity facilitated by the models of equal groups (Lampert, 1986), rectangular array and area (Izsák, 2004), and the procedure of repeated addition (Ambrose et al., 2003; Schifter et al., 2008; Schliemann et al., 1998). These contradictory findings regarding young students' understanding and use of distributivity can partly be explained by the methodology of the studies; students might perform better, for example in a teaching unit (Lampert, 1986), compared to a testing situation (Squire et al., 2004), but it is still unclear whether distributivity evolves easily or not. Another interpretation is that implicit use of distributivity is different from explicit understanding of the property (Ding & Li, 2014). The