Journal of Mathematical Behavior 45 (2017) 167–182

Contents lists available at ScienceDirect

The Journal of Mathematical Behavior

journal homepage: www.elsevier.com/locate/jmathb

Graphing formulas: Unraveling experts' recognition processes

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ARTICLE INFO

Article history:
Received 10 June 2016
Received in revised form 12 January 2017
Accepted 25 January 2017
Available online 9 February 2017

Keywords:
Graphing formulas
Experts' recognition
Function families
Prototypes and attributes

ABSTRACT

An instantly graphable formula (IGF) is a formula that a person can instantly visualize using a graph. These IGFs are personal and serve as building blocks for graphing formulas by hand. The questions addressed in this paper are what experts' repertoires of IGFs are and what experts attend to while recognizing these formulas. Three tasks were designed and administered to five experts. The data analysis, which was based on Barsalou and Schwarz and Hershkowitz, showed that experts' repertoires of IGFs could be described using function families that reflect the basic functions in secondary school curricula and revealed that experts' recognition could be described in terms of prototype, attribute, and part-whole reasoning. We give suggestions for teaching graphing formulas to students.

1. Introduction

Algebraic concepts, like functions, can be explored more deeply through linking different representations (Duval, 2006; Heid, Thomas, & Zbiek, 2012). Graphs and algebraic formulas are important representations of functions. Graphs seem to be more accessible than formulas (Leinhardt, Zaslavsky, & Stein, 1990; Moschkovich, Schoenfeld, & Arcavi, 1993). In addition, graphs give more direct information on covariation, that is, how the dependent variable changes as a result of changes of the independent variable (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). A graph shows features such as symmetry, intervals of increase or decrease, turning points, and infinity behavior. In this way, it visualizes the "story" that an algebraic formula tells. Therefore graphs are important in learning algebra, in particular in learning to read algebraic formulas (Eisenberg & Dreyfus, 1994; Kieran, 2006; Kilpatrick & Izsak, 2008; NCTM, 2000; Sfard & Linchevski, 1994).

Students have difficulties in seeing a function both as an input-output machine and as an object (Ayalon, Watson, & Lerman, 2015; Gray & Tall, 1994; Oehrtman, Carlson, & Thompson, 2008; Sfard, 1991). Graphs appeal to a gestalt-producing ability, and in this way can help to consolidate the functional relationship into a graphical entity (Kieran, 2006; Moschkovich et al., 1993). Graphs are also considered important in problem solving. Graphs are used for understanding the problem situation, recording information, exploring, and monitoring and evaluating results (Polya, 1945; Stylianou & Silver, 2004).

So, the ability to switch between representations, representation versatility, in particular conversions from algebraic formulas to graphs, is important in understanding algebra and in problem solving (Duval, 2006; NCTM, 2000; Stylianou, 2011; Thomas, Wilson, Corballis, Lim, & Yoon, 2010).

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http://dx.doi.org/10.1016/j.jmathb.2017.01.002

In a previous study a framework was developed to describe strategies for graphing formulas without using technology (Kop, Janssen, Drijvers, Veenman, & Van Driel, 2015). In the framework, it is indicated how recognition guides heuristic search. When one has to graph a formula there are different possible levels of recognition: from complete recognition (one immediately knows the graph) to no recognition at all (one does not know anything about the graph). For every level of recognition the framework provides strong to weak heuristics.

For the two highest levels of recognition the graph is completely recognized or the formula is recognized as a member of a function family whose graph characteristics are known. For instance, at the highest level of recognition the graph of $y = x^2$ is instantly recognized as a parabola with minimum (0,0). At the second level of recognition, $y = 4{\cdot}0.75^x + 3$ is recognized as a member of the family of decreasing exponential functions, and so the horizontal asymptote is read from the formula. In this way the graph can be instantly visualized. Another example at this level: $y = -x^4 + 6x^2$ is recognized as a polynomial function of degree 4; because of the negative head coefficient its graph has an M-shape or an Λ-shape; a short investigation of, for instance, the zeroes will instantly give the graph.

At these two highest levels of recognition in the framework, formulas can be instantly linked to graphs. Therefore, these formulas are defined as *instantly graphable formulas* (IGF). A large set of IGFs is beneficial to proficiency in graphing formulas. The current study was focused on experts' recognition processes when dealing with IGFs. For this study we defined an expert as a person with at least a master's degree in mathematics and at least 10 years of experience teaching at the secondary or college level, with experience in graphing formulas by hand. Although these experts are expected to be able to instantly link many formulas to graphs, their repertoires of IGFs remain unknown. In addition, we investigated what experts attend to when recognizing IGFs. This information might give suggestions for a repertoire of IGFs for students and for a focus in teaching students IGFs.

2. Theory

2.1. Cognitive units as building blocks

IGFs can be seen as building blocks in thinking and reasoning with and about formulas and graphs. Barnard and Tall (1997) introduced the concept of "cognitive unit", an element of cognitive knowledge that can be the focus of attention altogether at one time. For experts, well-connected cognitive units can be compressed into a new single cognitive unit which can be used as just one step in a thinking process (Crowley & Tall, 1999). In this way experts' knowledge is well organized in hierarchical mental networks with complex cognitive units, which can be enlisted when necessary (Campitelli & Gobet, 2010; Chi, Feltovich, & Glaser, 1981; Chi, 2011).

As IGFs are cognitive units in graphing formulas, they can be combined (addition, multiplication, chaining, etc.) and can form new, more complex IGFs. For instance, when dealing with $y = -x^4 + 6x^2$, novices may recognize the IGFs $y = -x^4$ and $y = 6x^2$ and have to combine these two IGFs to draw a graph, whereas $y = -x^4 + 6x^2$ is an IGF for experts, who recognize a 4th degree polynomial function. For experts, a formula like $y = x^2 - 6x + 5$ can trigger other cognitive units, like "its graph is a parabola with a minimum value", and the equivalent formulas $y = (x-1)(x-5)$ and$y = (x-3)^2 - 4$, which can give information about the zeroes and the minimum value, etc. Experts are expected to have more, and more complex, IGFs than novices, which generally enable them to graph formulas with fewer demands on the working memory (Sweller, 1994).

The current study was focused on recognition: in particular, which formulas and/or function families were instantly recognized by experts and how the recognition processes can be described.

2.2. Recognition described using Barsalou's model with prototype, attribute, and part-whole reasoning

Barsalou (1992) showed how human knowledge is organized in categories or concepts. People construct these categories based on attributes. When a task requires a distinction to be drawn between exemplars of a category, people construct new attributes and in this way new categories (Barsalou, 1992). For instance, for the concept bird, attributes (variables) like size, color, and beak, with several values, can be used to distinguish different exemplars. Categories can have a large diversity of exemplars, but have a graded structure (Eysenck & Keane, 2000; Barsalou, 2008). Some exemplars in a category are more central to that category than others; these are called prototypes. For instance, a robin is considered a more typical example of a bird than, for instance, a chicken or a penguin. When dealing with exemplars of a category, people tend to associate prototypical features with these exemplars (Barsalou, 2008; Schwarz & Hershkowitz, 1999). The tendency to reason from prototypes can pose problems. Since concept formation is not necessarily done using pure definitions, Watson and Mason (2005) emphasized the need to go beyond prototypes and to search for the boundaries of a concept. In this way one becomes aware of the dimensions of possible variation and in each dimension of the range of permissible change (Bills et al., 2006; Sandefur, Mason, Stylianides, & Watson, 2013; Watson & Mason, 2005). The personal example space, the collection of examples and the interconnection between the examples a person has at his/her disposal (the accessible example space), play a major role in how a person makes sense of the tasks he/she is confronted with (Watson & Mason, 2005; Goldenberg & Mason, 2008). Vinner and Dreyfus (1989) used concept image to emphasize the personal character of people's mental networks. These concept images determine what a person "sees" when dealing with concepts or categories, and are used in rapid identification.