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Probability: From classical to fuzzy ☆

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Abstract

The fuzzification of the classical probability has been initiated by L.A. Zadeh: the classical crisp random events are extended to measurable fuzzy random events and the probability measures are extended to probability integrals. The corresponding fuzzy probability (and fuzzy statistics) can be viewed as an integral part of soft computing. We follow a different approach to fuzzification developed primarily by S. Gudder and S. Bugajski: the classical notion of random variable is modified so that new theory models both fuzzy and quantum phenomena. Our goal is to survey the second approach, GB-approach, to the fuzzification of classical probability using elementary category theory. The motivation comes primarily from quantum physics: to some crisp outcome in the sample space there corresponds a whole spectrum (a genuine probability measure) in the state space; the same situation appears in other fields of science (expert systems, fuzzy logic), too. The corresponding generalization of a random variable constitutes a stochastic channel connecting two objects. Due to the duality, the channel can be equivalently described via generalized random variables and observables. In the fuzzified probability theory, an observable can map a crisp random event to a genuine fuzzy random event. First, in a bottom-up style, we recall some constructions and point out some properties of basic notions in GB-approach. Second, in a top-down style, we outline a simple model of fuzzified probability and show that the properties suffice to build the model so that it can be seen as a minimal extension of the classical probability theory. Fuzzy random events become Łukasiewicz logic propositional functions, probability measures and other relevant maps become morphisms, and basic constructions can be described via commutative diagrams.

Keywords: Fuzzification of classical probability theory; Probability domain; Probability integral; D-poset; Łukasiewicz tribe; Stochastic channel; Observable; Statistical map; Duality; Epireflection

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1. Introduction

Our goal of is to survey the fuzzification of classical probability theory as developed by S. Gudder ([20]) and S. Bugajski ([2,3]) using elementary category theory. The main contribution is the presentation of fuzzified probability as an epireflection of the classical one. Observables constitute morphisms and probability integrals as the fuzzification of probability measures are special morphisms into the fuzzification of the trivial (two-element) Boolean algebra. Fuzzy random events are propositional functions in the Łukasiewicz multivalued logic.

In his pioneering paper [32], L.A. Zadeh has proposed to consider Borel fuzzy sets as fuzzy random events and probability integral of the corresponding measurable functions as their probabilities. Having in mind "soft computing" applications, he proposed fuzzy operations on fuzzy random events and introduced basic fuzzified probability notions (cf. [33]). More information about the resulting fuzzy probability and applications can be found at http://people.eecs.berkeley.edu/~zadeh/papers. Zadeh's principal thesis is that (a) probability theory has a sufficiently rich structure for incorporating fuzzy sets within its framework and (b) probability theory and fuzzy set theory can be made to work in concert.

A thorough analysis of operations on fuzzy random events and generalized probability measures on fuzzy random events (based on T-norms and T-conorms) has been provided by R. Mesiar in [24], by B. Riečan and T. Neubrunn in [31], and by M. Navara in [25]. Collections of fuzzy random events can be viewed as quantum structures (effect algebras, D-posets). A comprehensive treatise on quantum structures [7] by A. Dvurečenskij and S. Pulmannová serves as a standard reference.

It seems justified to call and understand by the fuzzy probability what has resulted from Zadeh's approach and to find another name for the fuzzified probability motivated by quantum phenomena, as presented by S. Gudder in [20] and S. Bugajski in [2,3]; while Gudder sticks with fuzzy probability, Bugajski has proposed to call it operational probability.

The theory as outlined by Gudder and Bugajski (many other authors have contributed to, see [2,3]) rests heavily on deep theorems of abstract analysis. We believe that basic category theory apparatus provides tools and language to reformulate and prove fundamental theorems and to present the resulting probability in a more transparent way, cf. [9,12–15,17].

Section 2 is devoted to the motivation and mathematical background.

In Section 3 we recall some basic definitions and present some basic constructions of the fuzzified probability. It is written in a "bottom-up" manner. Theorems point out certain properties of maps (states, observables) which will serve as the base of fuzzified probability theory. As measure theory and abstract analysis served A.N. Kolmogorov to upgrade probability theory, a categorical approach should lead to an upgrading of the Kolmogorovian theory (cf. [6]).

In Section 4, written in a "top-down" manner, we outline a simple model of fuzzified probability based on observables and present some examples. We believe that the theorems and constructions in [20,2,3,9,12–15,17] point to "observables" as the morphisms of fuzzified probability theory. Observe that probability measures and probability integrals (states) become particular cases of observables.

2. Motivation

Besides L.A. Zadeh's motivation based on soft computing and the motivation based on quantum physics phenomena, there is another reason to fuzzify the classical probability theory (as founded by A.N. Kolmogorov in [21]): to develop a categorical extension, as simple as possible, so that probability measures become morphisms, basic constructions can be represented via commutative diagrams, and some implicit relationships become explicit.

In our opinion, the fuzzification of Kolmogorov's model to capture quantum physics phenomena has been very well presented by S. Gudder in [20] and S. Bugajski in [2,3]. Besides giving motivating examples, the authors have explained the usefulness of switching from Boolean type random events to fuzzy random events, represented by effect algebras of measurable functions, and have developed mathematical tools to prove theorems which generalize the classical ones.

To stress the applicability of the fuzzified model, let us quote the abstract of [3]. "The concept of operational random variable generalizing that of random variable is discussed and shown to be implied by the operational description of measurement. It is proved that any family of operational random variables having independent outcomes can be well represented by a single standard random variable. Nevertheless, it is demonstrated that some families of operational