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Fuzzy neighborhood operators based on fuzzy coverings

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Abstract

In many data mining processes, neighborhood operators play an important role as they are generalizations of equivalence classes which were used in the original rough set model of Pawlak. In this article, we introduce the notion of fuzzy neighborhood system of an object based on a given fuzzy covering, as well as the notion of the fuzzy minimal and maximal descriptions of an object. Moreover, we extend the definition of four covering-based neighborhood operators as well as six derived coverings discussed by Yao and Yao to the fuzzy setting. We combine these fuzzy neighborhood operators and fuzzy coverings and prove that only sixteen different fuzzy neighborhood operators are obtained. Moreover, we study the partial order relations between those sixteen operators.

Keywords: Fuzzy coverings; Fuzzy neighborhood operators; Rough sets

1. Introduction

Rough sets were introduced by Pawlak as a methodology for data analysis based on the approximation of concepts in data tables. It handles uncertainty in information systems due to indiscernibility and incompleteness. In the original model of Pawlak [17], an equivalence relation E is used to describe the indiscernibility among pairs of objects of a universe U and to define an approximation space. On the other hand, it is well-known that an equivalence relation can be equivalently specified by a partition of the universe, thus an approximation space can also be formulated in terms of a partition. In [28], Yao and Yao observe that still another equivalent structure can be considered to define the same approximation space, the σ-algebra of subsets of the universe whose atoms are the equivalence classes and whose elements are unions of these equivalence classes. Yao and Yao refer to these three equivalent structures as the element-based, the granule-based and the subsystem-based definitions of Pawlak's model.

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From an application perspective, working with equivalence relations is often a too strong assumption in order to obtain useful results. Therefore, Pawlak's model has been generalized along each of the three mentioned formulations. Indeed, by replacing the equivalence relation in the element-based definition by a general binary relation, or equivalently by a neighborhood operator, a first generalization is obtained [21,23,26,27]. In this case, the binary relation or the neighborhood operator determines collections of sets which no longer form a partition of U. A second generalization is obtained when we replace, in the granule-based definition, a partition by a covering, i.e., by a collection of non-empty sets such that its union is equal to U [18,27,31]. Finally, a third kind of generalized models is obtained when in the subsystem-based definition we replace the σ-algebra of subsets by a pair of systems: a closure system over U, i.e., a family of subsets of U that contains U and is closed under set-intersection, and its dual system [28]. However, these different generalizations are no longer equivalent, yielding different generalized rough set models.

For instance, in [28], given a covering $\mathbb{C}$, it is shown how one can derive four neighborhood operators as well as six other coverings, two of which coincide. The resulting twenty-four neighborhood operators are analyzed in [6] and reduced to thirteen groups of neighborhood operators. Moreover, the Hasse diagram of the thirteen neighborhood operators is obtained. Six of the considered neighborhood operators result in new covering-based rough set approximation operators, which are included in the framework of [20]. Furthermore, the connection between different covering-based approximation operators and relation-based approximation operators is discussed.

Both element-based and granule-based rough set models have been extended to the fuzzy setting in order to deal with real-valued data [4,5]. In these extensions, a key concept is that of a fuzzy neighborhood operator. In the literature, fuzzy neighborhood operators are often used in the context of fuzzy topology, e.g., [12,14–16], in order to describe concepts such as open and closed sets, and interior and closure operators. Interior and closure operators are closely related with the concept of approximation operators in data analysis, and here we focus on the concept of a fuzzy neighborhood operator from the perspective of fuzzy rough set theory, which is a hybridization of fuzzy set theory [30] and rough set theory [17]. The combination of both theories yields massive potential for information systems with real-valued data. The fuzzy neighborhood operators discussed in this article can be used to define fuzzy rough approximation operators [4,24].

In the last couple of years, initial efforts have been done to extend covering-based rough set models to the fuzzy setting [5,13,9,10,22]. Continuing the fuzzification of rough sets models, we extend here the definitions of the twenty-four neighborhood operators of [28] to the fuzzy setting, and we discuss equalities and partial order relations between them, extending the analysis done in [6] for the crisp case.

The outline of the article is as follows. In Section 2, we discuss some preliminary results concerning crisp neighborhood operators based on crisp coverings. In Section 3, different fuzzy neighborhood operators based on a fuzzy covering are introduced. Moreover, the properties of the fuzzy neighborhood operators are discussed. Additionally in Section 4, six fuzzy coverings derived from one fuzzy covering are studied. In Section 5, we discuss equalities between different fuzzy neighborhood operators based on a finite fuzzy covering and in Section 6, partial order relations between them are identified in order to obtain the Hasse diagram. Finally, we state some conclusions and future work in Section 7.

2. Preliminaries

Throughout this paper we assume that the universe U is a non-empty set. We start by discussing the relevant concepts in the crisp setting.

A *neighborhood operator* [28] on the universe U is a mapping $N\colon U \to \mathscr{P}(U)$, where $\mathscr{P}(U)$ represents the collection of subsets of U. In general, we assume that a neighborhood operator N is reflexive, i.e., $x \in N(x)$ for each $x \in U$. Moreover, a neighborhood operator is called symmetric if for all $x, y \in U$ it holds that $x \in N(y)$ if and only if $y \in N(x)$, and it is called transitive if for all $x, y \in U$ it holds that $x \in N(y) \Rightarrow N(x) \subseteq N(y)$.

Given a universe U and a collection $\mathbb{C} = \{K_i \subseteq U : K_i \neq \emptyset, i \in I\}$ of non-empty subsets of U, with I an index set, $\mathbb{C}$ is called a *covering* of U if $\bigcup_{i\in I} K_i = U$ [32]. A covering $\mathbb{C}$ induces the following neighborhood system for $x \in U$ [28]:

$$\mathscr{C}(\mathbb{C}, x) = \{K \in \mathbb{C} : x \in K\}. \tag{1}$$