ARTICLE IN PRESS
FSS:7044
Available online at www.sciencedirect.com
ScienceDirect
Fuzzy Sets and Systems ••• (••••) •••–•••
FUZZY sets and systems
www.elsevier.com/locate/fss

Short Communication

A note on the Sobol' indices and interactive criteria

Michel Grabisch [a,*], Christophe Labreuche [b]

[a] Paris School of Economics, University of Paris I, 106-112, Bd de l'Hôpital, 75013 Paris, France
[b] Thales Research and Technology, 1, Avenue Augustin Fresnel, 91767 Palaiseau, France

Received 8 January 2016; received in revised form 9 May 2016; accepted 14 May 2016

Abstract

The Choquet integral and the Owen extension (or multilinear extension) are the most popular tools in multicriteria decision making to take into account the interaction between criteria. It is known that the interaction transform and the Banzhaf interaction transform arise as the average total variation of the Choquet integral and multilinear extension respectively. We consider in this note another approach to define interaction, by using the Sobol' indices which are related to the analysis of variance of a multivariate model. We prove that the Sobol' indices of the multilinear extension give the square of the Fourier transform, a well-known concept in computer sciences. We also relate the latter to the Banzhaf interaction transform and compute the Sobol' indices for the 2-additive Choquet integral.

Keywords: Capacity; Interaction index; Sobol' index; Multilinear extension; Fourier transform

1. Introduction

In multicriteria decision making (MCDM), the Choquet integral with respect to a capacity has become a popular tool to model situations where some interaction exists between criteria [7]. This often happens in practice, as the evaluation of an alternative under several criteria is a complex process, where the level of importance of criteria generally depends on which criteria are satisfied or not. Let us give an example.

Example 1. The design of a complex system or software requires making trade-offs among multiple and conflicting concerns, such as safety, security, power consumption, communication overhead, components lifetime and costs for a Secure Radio Architecture [12]. A multi-criteria decision making approach is proposed in [12] in order to help finding the best compromise among these concerns seen as criteria. These criteria interact one another. For instance, safety and security can be considered as veto since the customer cannot accept an alternative not fulfilling these criteria. Criteria power consumption and communication overhead are complementary (positive interaction) as satisfying only one of them decreases drastically the quality of the alternative, according to the Secure Radio architect.

* Corresponding author.
E-mail addresses: michel.grabisch@univ-paris1.fr (M. Grabisch), christophe.labreuche@thalesgroup.com (C. Labreuche).

http://dx.doi.org/10.1016/j.fss.2016.05.009
0165-0114/© 2016 Elsevier B.V. All rights reserved.

Please cite this article in press as: M. Grabisch, C. Labreuche, A note on the Sobol' indices and interactive criteria, Fuzzy Sets Syst. (2016), http://dx.doi.org/10.1016/j.fss.2016.05.009

The basic ingredient to model interaction is the capacity (or fuzzy measure) defined on the set N of criteria, through its interaction transform [5], which is a generalization of the Shapley value. Another type of interaction has been introduced by Roubens [15] under the name *Banzhaf interaction transform*, and extends the Banzhaf value. So far, emphasis has been put on the former one in theoretical developments and applications.

It is well known that the Choquet integral is an extension of a capacity, seen as a pseudo-Boolean function, and called the Lovász extension. Another popular extension of pseudo-Boolean functions is the Owen extension or multilinear extension, known in Multi-Attribute Utility Theory from a long time ago [10]. Both can be considered as aggregation functions on a bounded closed domain (say, $[0, 1]^n$).

In applications, interpretation of models and their sensitivity analysis is important for the user. The following example illustrates that several sensitivity analyses can be necessary.

Example 2 *(Example 1 cont.).* Two problems arise in the design of a Secure Radio Architecture [12]. First of all, the engineering process is an incremental process in which an initial architectural alternative is assessed, and given its evaluation, the architect has to decide on improvement means to apply to this alternative. She needs thus to know on which criteria, an improvement is (on average) the most influential on the overall score.

Another problem is that the values of the attributes are obtained from dedicated analysis or simulation tools that can be time consuming. For instance, safety can be assessed by fault tree analysis techniques. It can therefore happen that the assessments of some alternatives on some criteria are missing, because of time constraints. The architect needs to identify the criteria for which a missing value has the largest impact on the uncertainty of the global evaluation.

So far, in MCDM, the two issues raised in Example 2 are handled by the same concept of mean importance and interaction indices. The interaction index for $S \subseteq N$ w.r.t. an aggregation function F on $[0, 1]^n$ can be defined as the total variation of F w.r.t. the coordinates in S – the importance of a criterion being a particular case when S is a singleton. Grabisch et al. [8] showed that the interaction transform corresponds to the interaction index w.r.t. the Choquet integral, which generalizes the Shapley value, while the Banzhaf interaction transform is the interaction index w.r.t. the multilinear extension. We stress in this note that the interaction index is, by its definition, a relevant tool for sensitivity analysis in the sense of the first issue raised in Example 2. However it is not suited for the second one.

To this end, we need to adopt another view of interaction, namely the statistical view. We start by considering a well-known transform used in computer sciences but so far ignored in the field of decision making, which is the Fourier transform. ANOVA (analysis of variance) is an interesting approach to decompose a multivariate function into a sum of terms, each depending on a subset of the set of variables. The decomposition is orthogonal in the sense that each term has zero mean, except for the empty set. Then the variance of the function is decomposed as the variance of each term, which yields the Sobol' indices. It is thus a natural way to deal with the second issue in Example 2. We show that considering the aggregation model as a multivariate function and defining the interaction index of $S \subseteq N$ through the Sobol' index of S, we come up with the square of the Fourier transform when the aggregation function is the multilinear model (Theorem 1). We also show the close relation between the Fourier transform and the Banzhaf interaction transform, and compute the Sobol' indices for the 2-additive Choquet integral (Theorem 2). The result for the Choquet integral is limited to 2-additive capacities as there does not seem to be a simple and compact expression for a general capacity. This case is of particular importance in practice, since it constitutes a good compromise between versatility and complexity. Experimental studies in multicriteria evaluation have shown that 2-additive capacities have almost the same approximation ability than general capacities (see, e.g., [6]). Most of the applications of the Choquet integral are restricted to a 2-additive capacity [12,1,2].

Apart from the distinction between the two issues depicted in Example 2, the Sobol' indices have the advantage to be comparable. One can say for instance that the influence of the interaction between criteria 1 and 2 is larger than that of the importance of criterion 1 taken alone. Such a comparison is not relevant with the interaction indices.

Throughout the paper, cardinalities of sets $S, T, \ldots$ are denoted by the corresponding small letters $s, t, \ldots$.

2. Basic notions

We consider throughout a finite set $N = \{1, \ldots, n\}$.

Please cite this article in press as: M. Grabisch, C. Labreuche, A note on the Sobol' indices and interactive criteria, Fuzzy Sets Syst. (2016), http://dx.doi.org/10.1016/j.fss.2016.05.009