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On comonotone commuting and weak subadditivity properties of seminormed fuzzy integrals

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Abstract

In the paper, we characterize the class of all the binary operators $\circ$-commuting with the generalized Sugeno integral generated by a strict t-norm. For the Sugeno integral, this problem was solved by Ouyang and Mesiar. We also present the necessary and sufficient conditions for the weak subadditivity of the generalized Sugeno integral.

Keywords: Generalized Sugeno integral; Seminorm; Capacity; Monotone measure; Semicopula; Fuzzy integrals; Shilkret integral

1. Introduction

Let $(X, \mathcal{A})$ be a measurable space, where $\mathcal{A}$ is a σ-algebra of subsets of a non-empty set X, and let $\mathcal{S}$ be the family of all measurable spaces. The class of all $\mathcal{A}$-measurable functions $f: X \to [0,1]$ is denoted by $\mathcal{F}_{(X,\mathcal{A})}$. A *capacity* on $\mathcal{A}$ is a non-decreasing set function $\mu: \mathcal{A} \to [0,1]$ with $\mu(\emptyset) = 0$ and $\mu(X) = 1$. We denote by $\mathcal{M}_{(X,\mathcal{A})}$ the class of all capacities on $\mathcal{A}$.

Suppose that $\mathrm{S}: [0,1]^2 \to [0,1]$ is a *semicopula* (also called a t-*seminorm*), i.e., a non-decreasing function in both coordinates with the neutral element equal to 1. It is clear that $\mathrm{S}(x, y) \leqslant x \wedge y$ and $\mathrm{S}(x, 0) = 0 = \mathrm{S}(0, x)$ for all $x, y \in [0,1]$, where $x \wedge y = \min(x, y)$ (see [1,3,7]). We denote the class of all semicopulas by $\mathfrak{S}$. There are three important examples of semicopulas: M, Π and S_L, where $\mathrm{M}(a,b) = a \wedge b$, $\Pi(a,b) = ab$ and $\mathrm{S}_L(a,b) = (a+b-1) \vee 0$ usually called the *Łukasiewicz t-norm* [7]. Hereafter, $a \vee b = \max(a,b)$.

The generalized Sugeno integral is defined by

$$\mathbf{I}_{\mathrm{S}}(\mu, f) := \sup_{t \in [0,1]} \mathrm{S}\Big(t, \mu\big(\{f \geqslant t\}\big)\Big), \tag{1}$$

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where $\{f \geqslant t\} = \{x \in X : f(x) \geqslant t\}$, $(X, \mathcal{A}) \in \mathcal{S}$ and $(\mu, f) \in \mathcal{M}_{(X,\mathcal{A})} \times \mathcal{F}_{(X,\mathcal{A})}$. In the literature the functional $\mathbf{I}_\mathrm{S}$ is also called the *seminormed fuzzy integral* [4,8,10]. Replacing semicopula S with M, we get the *Sugeno integral* [14]. Moreover, if $\mathrm{S} = \Pi$, then $\mathbf{I}_\Pi$ is called the *Shilkret integral* [13].

The aim of the paper is to give a partial answer to the problem of characterization of all the binary operators $\circ$ which are commuting with the generalized Sugeno integral under the condition of comonotonicity of functions f, g. We also discuss the problem of weak subadditivity of the generalized Sugeno integral (see [5]). In Section 2 we present our main results as well as some related results. We describe all binary operators commuting with the integral $\mathbf{I}_\mathrm{S}$ for a wide class of semicopulas and provide a solution to Problem 2.31, which was posed by Borzová-Molnárová, Halčinová and Hutník [5] (see also [9]). In Section 3 we give the necessary and sufficient conditions for weak subadditivity of the generalized Sugeno integral.

2. Comonotone commuting property of integral $\mathbf{I}_\mathrm{S}$

We say that $f, g : X \to [0, 1]$ are *comonotone* on $A \in \mathcal{F}$, if $\big(f(x) - f(y)\big)\big(g(x) - g(y)\big) \geqslant 0$ for all $x, y \in A$. Clearly, if f and g are comonotone on A, then for any real number t either $\{f \geqslant t\} \subset \{g \geqslant t\}$ or $\{g \geqslant t\} \subset \{f \geqslant t\}$. In what follows $\mathbb{1}_E$ stands for the indicator of $E \subset X$ and $h_E(x) = h(x)\mathbb{1}_E(x)$, $x \in X$. Ouyang and Mesiar [11] posed the following problem.

Problem 1. Let $\mathrm{S} \in \mathfrak{S}$ be fixed. Find all operators $\circ : [0, 1]^2 \to [0, 1]$ such that for any measurable space $(X, \mathcal{A})$, any capacity $\mu \in \mathcal{M}_{(X,\mathcal{A})}$ and any $E \in \mathcal{A}$

$$\mathbf{I}_\mathrm{S}\big(\mu, (f \circ g)_E\big) = \mathbf{I}_\mathrm{S}(\mu, f_E) \circ \mathbf{I}_\mathrm{S}(\mu, g_E) \tag{2}$$

for all comonotone functions $f, g : X \to [0, 1]$.

Under the assumption that $E = X$, this problem was also formulated by Borzová-Molnárová, Halčinová and Hutník [5], Problem 2.32. Mesiar and Ouyang [12] examined Problem 1 for the Sugeno integral

$$(S)\int f \,\mathrm{d}\mu = \sup_{t\in[0,\infty]} \big(t \wedge \mu(\{f \geqslant t\})\big). \tag{3}$$

They proved that if $\lim_{b\to\infty}(a \circ b) \in \{a, \infty\}$ and $\lim_{a\to\infty}(a \circ b) \in \{b, \infty\}$ for all a, b, then the integral (3) possesses the comonotone $\circ$-commuting property if and only if $\circ$ equals to one of the four operators: $\wedge, \vee$, PF and PL, where PF and PL are the first and last projection, respectively, that is, $\mathrm{PF}(a, b) = a$ and $\mathrm{PL}(a, b) = b$ for all a, b. This result is also true for the integral (1) with $\mathrm{S} = \mathrm{M}$ provided that

C1. $\lim_{b\to 1}(a \circ b) \in \{a, 1\}$, $\lim_{a\to 1}(a \circ b) \in \{b, 1\}$ for all $a, b \in [0, 1]$

(see [12], p. 458). Moreover, Ouyang and Mesiar [11] proved that there are only 18 operators, including the minimum, the maximum, the first projection and the last projection, such that (2) with $\mathrm{S} = \mathrm{M}$ holds.

In this paper, we examine Problem 1 for an arbitrary semicopula S. Firstly, we give the necessary condition for the commuting property.

Proposition 1. *Fix a semicopula* S. *If the integral* $\mathbf{I}_\mathrm{S}$ *has the property* (2), *then*

$$a \circ b = \begin{cases} \mathrm{S}(e_2, b) \vee \mathrm{S}(e_3, a) & \text{for } a \leqslant b \\ \mathrm{S}(e_1, a) \vee \mathrm{S}(e_3, b) & \text{for } a > b \end{cases} \tag{4}$$

with numbers e_i *being such that* $0 \leqslant e_i \leqslant e_3 \leqslant 1$ *for* $i = 1, 2$. *Moreover,* $e_1 = 1 \circ 0$, $e_2 = 0 \circ 1$ *and* $e_3 = 1 \circ 1$.

Proof. Putting $f = g = \mathbb{1}_E$ in (2), we get $\mathrm{S}\big(1 \circ 1, \mu(E)\big) = \mu(E) \circ \mu(E)$. For $\mu(E) = 0$, we obtain $0 = 0 \circ 0$. Next, we shall prove that $0 \circ 1 \leqslant 1 \circ 1$. In fact, suppose that $1 \circ 1 < 0 \circ 1$ and suppose A is a non-empty proper subset of B. Setting $E = X$, $f = \mathbb{1}_A$ and $g = \mathbb{1}_B$ in (2) yields

$$\mathrm{S}\big(1 \circ 1, \mu(B)\big) \vee \mathrm{S}\big(0 \circ 1, \mu(B \backslash A)\big) = \mu(A) \circ \mu(B).$$

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