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Some properties of idempotent uninorms on a special class of bounded lattices

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ABSTRACT

Binary operations have many applications in fuzzy set theory. One of them is uninorms. In this paper, we study some properties of idempotent uninorms on bounded lattices. It is shown that idempotent uninorms on an arbitrary bounded lattice need not always be internal (with the extended definition of the term ”internal”). We give a sufficient condition for the bounded lattice to any idempotent uninorm be internal. In addition that it is given many properties of idempotent uninorms in this case.

1. Introduction

Uninorms were introduced in [27] and investigated the structure of them in [29] by Yager and Rybalov. The structure of these operators in considerable detail was also studied by many authors in other papers [11,12,18]. Uninorms are a generalization of triangular norms (t-norms, for short) and triangular conorms (t-conorms, for short) that were studied in [26,28] with more applications. Uninorms allow the freedom for the neutral element e (sometimes called identity) to be an arbitrary element from unit interval [0, 1], which is 1 for t-norms and 0 for t-conorms. For more details on t-norms and t-conorms, we refer to [1,2,10,22]. If the neutral element of uninorm aggregation operators on [0, 1] is not 0 or 1, the construction of these operators is an important work. Uninorms play an important role not only in theoretical investigations but also in practical applications such as the expert systems, neural networks, fuzzy logics, etc.

Martin, Mayor and Torrens [23] characterized idempotent uninorms on unit interval [0, 1] as improvement of a well-known theorem of Czogała and Drewniak [6] on idempotent, associative and increasing operations with a neutral element. Associative, monotonic, idempotent operations with a neutral element are special combinations of minimum and maximum and, consequently, locally internal. The characterization of idempotent uninorms given in [11] is also locally internal. Two important classes of uninorms on unit interval [0, 1] are characterized, corresponding to the use of the minimum operator (the class $U_{\min}$) and maximum operator (the class $U_{\max}$). They are well-known examples of idempotent uninorms, which are the smallest and the largest idempotent uninorms with a neutral element.

De Baets et al. [12] characterized all idempotent uninorms defined on a finite ordinal scale, similar as on the unit interval the characterization of idempotent uninorms [11,23] and draws upon the work of Czogała and Drewniak [6]. Furthermore, it was proved [12] that any idempotent uninorm (called discrete uninorm) is uniquely determined by a decreasing function

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from the set of scale elements not greater than the neutral element to the set of scale elements not smaller than the neutral element. Some other studies related to uninorms can be found also in [10,13–17,19,21,24].

In this paper, after some preliminaries concerning locally internal operations on unit interval [0, 1], we study many properties of idempotent uninorms on the bounded lattices with some additional constraints. And it is shown that idempotent uninorms on an arbitrary bounded lattice need not always be internal (with the extended definition of the term "internal"). We give a sufficient condition for the bounded lattice to any idempotent uninorm be internal. In addition that it is given many properties of idempotent uninorms in this case.

2. Preliminaries

We recall here some definitions and results about binary operations on [0, 1] that are monotonic (increasing in each place) and satisfy the locally internal property, i.e. the value of such operations at any point (x, y) is always one of its arguments. Next, we consider uninorms on bounded lattices and their basic properties.

Definition 1 ([23]). A binary operation $F : [0, 1]^2 \to [0, 1]$ is locally internal if F satisfies the condition that $F(x, y) \in \{x, y\}$ for all $x, y \in [0, 1]$.

Lemma 1 ([23]). *Let F be a locally internal operation. For any $a, b, c \in [0, 1]$, we have $F(a, F(b, c)) = F(F(a, b), c)$ if and only if not all the values $F(a,b)$, $F(a,c)$ and $F(b,c)$ are different.*

Lemma 2 ([23]). *Let F be a locally internal, monotonic operation and $a, b, c \in [0, 1]$ such that the restriction of F to $\{a, b, c\}$ is commutative. Then $F(a, F(b, c)) = F(F(a, b), c)$.*

The following result is an immediate consequence of the Lemma 1 and Lemma 2 and shows the relationship between commutativity and associativity for locally internal, monotonic operations.

Proposition 1 ([23]). *If a locally internal, monotonic operation is commutative, then it is associative.*

Proposition 2 ([6]). *Idempotent, associative, monotonic operation with a neutral element is locally internal.*

Definition 2 ([4]). A lattice $(L, \leq)$ is bounded lattice if L has the top and bottom elements, which are written as 1 and 0, respectively, that is, there exist two elements $1, 0 \in L$ such that $0 \leq x \leq 1$, for all $x \in L$.

Definition 3 ([4]). A lattice $(L, \leq)$ is distributive lattice if the following two equivalent conditions hold:

i) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all $x, y, z \in L$,
ii) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all $x, y, z \in L$.

Definition 4 ([4]). Given a bounded lattice $(L, \leq, 0, 1)$ and $a, b \in L$, if a and b are incomparable, we use the notation $a \| b$.

Definition 5 ([4]). Given a bounded lattice $(L, \leq, 0, 1)$ and $a, b \in L$, $a \leq b$, a subinterval $[a, b]$ of L is defined as

$$[a, b] = \{x \in L \mid a \leq x \leq b\}.$$

Similarly, we define $]a, b] = \{x \in L \mid a < x \leq b\}$, $[a, b[= \{x \in L \mid a \leq x < b\}$ and $]a, b[= \{x \in L \mid a < x < b\}$.

Let $(L, \leq, 0, 1)$ be a bounded lattice and $e \in L$. Let $A(e) = [0, e] \times [e, 1] \cup [e, 1] \times [0, e]$ and $I_e = \{x \in L \mid x \| e\}$.

Definition 6 ([9,20]). Let $(L, \leq, 0, 1)$ be a bounded lattice. Operation $U : L^2 \to L$ is called a uninorm on L (shortly a uninorm, if L is fixed) if it is commutative, associative, increasing with respect to both variables and there exist an element $e \in L$ such that $U(x, e) = x$ for all $x \in L$. The element e is called the neutral element of U.

We denote by $\mathcal{U}(e)$ the set of all uninorms on the bounded lattice L with the neutral element $e \in L$.

Definition 7 ([3,8]). Operation T: $L^2 \to L$ (S: $L^2 \to L$) is called a t-norm (t- conorm) if it is commutative, associative, increasing with respect to both variables and has a neutral element $e = 1$ $(e = 0)$.

Proposition 3 ([20]). *Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L \setminus \{0, 1\}$ and U be a uninorm on L with the neutral element e. Then*

i) $T = U|[0, e]^2 : [0, e]^2 \to [0, e]$ *is a t-norm on* $[0, e]$.
ii) $S = U|[e, 1]^2 : [e, 1]^2 \to [e, 1]$ *is a t-conorm on* $[e, 1]$.

Proposition 4 ([20]). *Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L \setminus \{0, 1\}$ and U be a uninorm on L with the neutral element e. The following properties hold:*

i) $x \wedge y \leq U(x, y) \leq x \vee y$ *for all* $(x, y) \in A(e)$,
ii) $U(x, y) \leq x$ *for* $(x, y) \in L \times [0, e]$,
iii) $U(x, y) \leq y$ *for* $(x, y) \in [0, e] \times L$,
iv) $x \leq U(x, y)$ *for* $(x, y) \in L \times [e, 1]$,
v) $y \leq U(x, y)$ *for* $(x, y) \in [e, 1] \times L$.