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Multistability and bifurcation in a delayed neural network

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ABSTRACT

In this paper, the dynamical behaviors including multistability and bifurcation of a delayed neural network system are investigated. It is shown that the system coexists sixteen stable states with their own domains of attraction. All stable states are determined by using a Lyapunov function. Interestingly, the system exhibits bistability and double cycles with only one equilibrium. The existence of the Hopf bifurcation is well studied. It is shown that the Hopf bifurcation occurs as the time delay reaches a certain value. Simulation experiments supported our theoretical results.

1. Introduction

Neural network is commonly referred as an information processing paradigm which has a great many interconnected units working together by some specific activation functions. It is derived from the way animal central nervous systems process information. Since the invention of the noted Hopfield neural network [1], neural network has received increasing concern as its significant applications in pattern recognition and machine learning. A series of meaningful research findings have been proposed [2–5].

As is known to all, neural network has abundant dynamical behaviors for its strong nonlinearity and high dimensionality from a system perspective. These behaviors include stability and multistability, multi-periodicity, bifurcation, chaos and hyperchaos. Multistability which means the coexistence of multiple steady states is the foundation of associative memory storage and pattern recognition in neural network [6,7]. The occurrence of multistability is related to the fact that the changes in initial conditions can switch the stable state of system. Each steady state in neural network stimulates a brain state with special perceptual object which can be selected by introducing control input with respect to initial condition [8]. As to single neuron, multistability means that there coexist different firing patterns, such as silence, spiking, regular, etc. If the neuron is imposed by the noise, it can switch between the different firing patterns [9,10]. In engineering, Morfu et al. use the multistability of the cellular network to extract the regions of interest of an image representing the radiography of a soldering [11]. In fact, there are some considerable researches about the multistability in neural network [12–15]. The more comprehensive and systematic results are presented by Cheng et al. which have shown that n-dimensional delayed neural networks can have up to 2^n independent stable states with their own basins of attraction [16,17]. Huang et al. proposed some new theoretical results on multistability of neural networks when neurons undergo self-excitation and second-order synaptic connectivity [18]. Wang and Chen studied a class of neural network systems with Mexican-hat-type activation functions and deduced some sufficient conditions of multistability in the systems [19]. Zeng et al. investigated the stability of multiple equilibria of neural networks with time-varying delays and concave–convex characteristics [20]. If the activation functions of the n-neuron neural network are concave or convex in k intervals, then the neural network have k^n stable equilibria. Kaslik and Sivasundaram considered the multistability of Hopfield neural networks with distributed delays impulsive effects by applying Lyapunov functionals and stability theory [21]. Li et al. investigated the multistability of genetic regulatory networks with sum regulatory schemes by using piecewise linear sectors [22]. Du and Xu established the sufficient conditions of the multistability and multiperiodicity of Cohen–Grossberg bidirectional associative memory neural

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networks with discontinuous activation functions [23]. Bifurcation is also one of the very important properties of neural network. A bifurcation occurs when a small change made to system parameter causes a topological change of the system. It is generally recognized that bifurcations divide into local bifurcation and global bifurcation. These two types of bifurcations in neural network have been reported in some literature recently [24–27].

Though there are some studies with regard to neural network, analysis of the dynamical behaviors of the neural network is still an important and challenging work. Many unknown nonlinear characteristics yet need to be discovered. The existence of multiple equilibria in neural network leads to multiple coexisting attractors, including multistability, multiperiodicity, multichaos, coexisting point and periodic attractors, etc. These dynamical behaviors are determined by the initial values for given system parameters. The bifurcation in neural network is related to changes in system parameters. System parameters can cause fundamental changes in system dynamical behaviors. In addition, there are only a few literature systematically investigating multistability in company with bifurcation. Both the system parameters and initial values are the key factors affecting the system performance. In this paper, we will consider the multistability and bifurcation together for a delayed neural network system with four neurons. More detailed analysis of the multistability is presented. For different system parameters, the system has a different number of equilibria. The different types of the coexisting attractors can be seen in the system for given different system parameters. Studies show that both bistability and double periodic attractors are appeared in this system when it has only one equilibrium point. Sixteen stable states and their own basins of attraction are presented in the system. Each stable state corresponds to a positively invariant set. The existence of Hopf bifurcation is also discussed. When time delay exceeds a certain value, the system appears a Hopf bifurcation which results in a limit cycle.

2. Model formulation

In 1989, Marcus and Westervelt first introduced time delay to the Hopfield neural network for considering the asynchronous interaction between neurons [28]. The general delayed Hopfield neural network with n coupled neurons is described by [29,30]

$$\dot{x}_i(t) = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} g_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_{ij})) + I_i, \quad i = 1, \ldots, n,$$

where g_j and f_j ($j = 1, \ldots, n$) are activation functions, and I_i is the external input. There are many literature on the stability of this system. However, our interests are primarily focused on the multiple coexisting attractors and bifurcation of the above system. For simplicity, we consider the four neurons delayed neural network system with self-connection and ring structure as its links are shown in Fig. 1. The mathematical model of the system is given by

$$\begin{cases} \dot{x}_1(t) = -x_1 + af(x_1(t-\tau_1)) + bf(x_2(t-\tau_2)), \\ \dot{x}_2(t) = -x_2 + af(x_2(t-\tau_1)) + bf(x_3(t-\tau_2)), \\ \dot{x}_3(t) = -x_3 + af(x_3(t-\tau_1)) + bf(x_4(t-\tau_2)), \\ \dot{x}_4(t) = -x_4 + af(x_4(t-\tau_1)) + bf(x_1(t-\tau_2)), \end{cases} \tag{1}$$

where x_1, x_2, x_3, and x_4 are state variables of neurons, a and b are the connection weights, and τ_1 and τ_2 are the time delays. The activation function $f(\xi) = \tanh\xi$ is a sigmoidal function which is bounded in region $(-1, 1)$.

The system (1) deduced from the general Hopfield neural network model represents a special ring coupled architecture of the neural network. The dynamical analysis of the system (1) can help us better reveal the complex dynamical behaviors of the general high-dimensional case. In the following part, we will investigate the multistability and the Hopf bifurcation of the system (1). The multistability represents the coexistence of multiple stable states in a system. For given system parameters, the system can perform multistability if the system has multiple equilibria. Each equilibrium point is a point attractor with its own domain of attraction. Therefore, the multistability is related to the initial conditions of the system. The Hopf bifurcation means that the system changes from stable states to unstable periodic states if the system parameters pass through a certain critical value. It is closely related to the parameters of the system. If the system has multiple equilibria, then the system may occur as multiple Hopf bifurcations with the variation of the system parameters and initial values. In a sense, both the multistability and Hopf bifurcation are local dynamic behaviors around the equilibria of the system. The bifurcation can generate or eliminate the stability of the system, and the multiple coexisting bifurcations can generate or eliminate the multistability of the system.

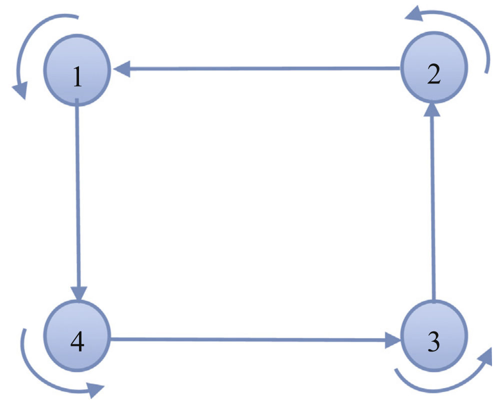

Fig. 1. The connections between neurons.

3. Multistability analysis

In this section, multistability in system (1) is presented by applying the geometric method of Cheng et al. [16,17]. It is shown that system (1) coexists sixteen stable states with respect to sixteen positively invariant sets. The situation of the coexistence of stable and periodic states is also investigated. The effectiveness of our theoretical results are verified by simulation experiments.

3.1. Analysis of the equilibria

Define $g(x_i) = -x_i + af(x_i(t)) + bf(x_j(t))$, $(i, j = 1, 2, 3, 4, i \neq j)$. It is clear that $g(x_i)$ is corresponding to the right side of system (1). Therefore, we only need to study the roots of $g(x_i) = 0$ for obtaining the equilibria of system (1). Define functions as follows:

$$\begin{aligned} g_+(u) &= -u + af(u) + |b|, \\ g_-(u) &= -u + af(u) - |b|. \end{aligned} \tag{2}$$

Obviously, $g_-(x_i) \le g(x_i) \le g_+(x_i)$ since function f is bounded in $(-1, 1)$. The derivatives of $g_\pm(u)$ are $g'_\pm(u) = -1 + af'(u)$.

If $a < 1$, $g'_\pm(u) < 0$ for $0 < f'(u) \le 1$, then $g_\pm(u)$ are monotonically decreasing as their plane graphs shown in Fig. 2. It follows that $g(x_i) = 0$ has one root. Consequently, system (1) has only one equilibrium point. In this case, system (1) may display stable and periodic motion.

If $a=1$, $g'_\pm(u) \le 0$, and $g'_\pm(u) = 0$ if and only if $u=0$. The