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Closure structures parameterized by systems of isotone Galois connections

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We study properties of classes of closure operators and closure systems parameterized by systems of isotone Galois connections. The parameterizations express stronger requirements on idempotency and monotony conditions of closure operators. The present approach extends previous approaches to fuzzy closure operators which appeared in analysis of object-attribute data with graded attributes and reasoning with if-then rules in graded setting and is also related to analogous results developed in linear temporal logic. In the paper, we present foundations of the operators and include examples of general problems in data analysis where such operators appear.

1. Introduction

In this paper we deal with closure structures which emerge in data-analytical applications such as formal concept analysis [31] of data with fuzzy attributes [14,40] and approximate reasoning such as inference of fuzzy if-then rules from data [8,9]. In particular, our paper generalizes and extends observations on fuzzy closure operators and related structures. Since their inception, fuzzy closure operators have been the subject of extensive research, the most influential early papers on the topic include [3,17,18,20,21,32,46,47]. As it is usual with graded (fuzzy) generalizations of classic notions, there are several sound ways to introduce closure operators in fuzzy setting. While most authors agree on the conditions of extensivity and idempotency, which take the same form as in the classic setting, the approaches differ in the treatment of the monotony condition. There are two major approaches:

1. Using the bivalent notion of inclusion of fuzzy sets, where the monotony condition can be written as "$A \subseteq B$ implies $\boldsymbol{c}(A) \subseteq \boldsymbol{c}(B)$" and means that "the closure of A is *fully contained* in the closure of B whenever A is *fully contained* in B." The full containment of fuzzy sets (here denoted "$\subseteq$") is defined as a bivalent relation on fuzzy sets so that for any fuzzy sets C and D in the universe X, we put $C \subseteq D$ whenever for each element $x \in X$, the degree to which x belongs to D is at least as high as the degree to which it belongs to C.
2. The second option uses a graded notion of inclusion of fuzzy sets. In this case, the monotony condition can be written as $\mathrm{S}(A, B) \leq \mathrm{S}(\boldsymbol{c}(A), \boldsymbol{c}(B))$, where $\leq$ is the order on the set of truth degrees (the usual order of reals if the scale of degrees is the real unit interval) and S is a suitable graded subsethood. Both $\mathrm{S}(A, B)$ and $\mathrm{S}(\boldsymbol{c}(A), \boldsymbol{c}(B))$ are general degrees of inclusion, i.e., $\mathrm{S}(A, B)$ is the degree to which A is included in B and analogously for $\mathrm{S}(\boldsymbol{c}(A), \boldsymbol{c}(B))$. Hence, the monotony condition can be read "the *degree* to which the closure of A is included in the closure of B is *at least as high* as the *degree* to which A is included in B." In other words, $\mathrm{S}(A, B)$ gives a lower bound of the inclusion degree of

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the closure of A in the closure of B. In the context of approximate inference, this is a desirable property because one can obtain a lower approximation of the degree $\mathrm{S}(\boldsymbol{c}(A), \boldsymbol{c}(B))$ without the need to actually compute either of $\boldsymbol{c}(A)$ and $\boldsymbol{c}(B)$.

These two basic approaches can be seen as two borderline requirements on the monotony condition and for reasonable choices of S, which includes the residuum-based fuzzy set inclusion proposed by Goguen [34], the second approach constitutes a stronger requirement than the first one. Interestingly, both the approaches can be handled by a single theory which leaves the approaches as special cases. In fact, there are several results on fuzzy closure operators where both the approaches result as special cases. The initial paper [4] by Belohlavek uses a general monotony condition which is parameterized by an order-filter on the set of truth degrees. Conceptually different approach has been introduced in [7] where the authors employ linguistic hedges, again, as parameters of the monotony condition. In both the approaches, the two basic notions of monotony result by chosen parameterizations—either a special filter in case of [4] or a special hedge in case of [7].

In our recent paper [54], we have developed a theory of graded if-then rules with general semantics parameterized by systems of isotone Galois connections. In this setting, general fuzzy closure operators with parameterized idempotency conditions appeared. Interestingly, in the approach to attribute implications with temporal semantics introduced first in [50] and developed further in [51], we have utilized conceptually similar structures which utilize the notion of being closed under “time shifts.” In this paper, we present results showing that most results related to closure structures in [54] and [51] can be handled by a single theory of closure structures defined on complete lattices which are parameterized by systems of isotone Galois connections. In addition, we also show that the approaches in [4,7] result as special cases of the presented formalism. Therefore, the present paper studies closure structures from the perspective of a general class of parameterizations, makes conclusions on a general level, and particular results like those in [4,7,51,54] can be obtained by selecting concrete parameterizations on complete lattices.

The closure structures studied in our paper generalize closure structures that have appeared in the analysis of clusters in fuzzy object-attribute data. Namely, they are closely related to structures of formal concepts which appear in formal concept analysis [31]. The input data for concept formal analysis can be seen as a collection of objects which are described by sets of their attributes. The primary interest of formal concept analysis is to extract particular clusters, called formal concepts, from the input data. In the classic setting, the input data is represented by a binary relation between a set of objects and a set of attributes/features, indicating whether objects have attributes/features. In graded extensions of the formal concept analysis [5], the input data is considered as a *fuzzy relation*, i.e., each object can have an attribute *to a degree* which is taken from a suitable scale of truth degrees. The graded extensions are motivated by the fact that attributes are often fuzzy (like high speed, low price,...) rather than crisp yes/no. From the computational point of view, it is important that formal concepts (i.e., the clusters extracted from data) are determined by fixed points of particular fuzzy closure operators since their computation can be reduced to enumeration of fixed points of a closure operator derived from the input data. This is one area where fuzzy closure operators related to those studied in our paper emerge. It has been shown in the past that focusing on variants of fuzzy closure operators may solve the problems of large sizes of collections of all clusters. Namely, if one uses a scale with k truth degrees, the number of extracted formal concepts from data can be (in the worst case) k^n where n is the size of the data. Although this extreme situation does not occur in practical situations, the number of extracted concepts from given data can grow dramatically. It is therefore necessary to look for methods of reducing the numbers of extracted clusters. One of the early methods utilizes *linguistic hedges* as additional constraints [14]. The method is based on a principle on decreasing logical precision based on choices of linguistic hedges which in turn ensures that fewer clusters are extracted from data. As a consequence, the constrained clusters can be determined by fixed points of a new type of fuzzy closure operators—fuzzy closure operators parameterized by hedges [7]. Such operators can also be seen as special cases of operators studied in our paper. From this point of view, we deal here with a family of closure structures which generalizes several types of closure structures that appear in formal concept analysis of data with fuzzy attributes. In Section 4, we present details on two important problems related to inference of if-then rules where operators parameterized by systems of isotone Galois connections also appear.

Our paper is organized as follows. In Section 2, we present a survey of notions related to closure operators and closure systems and introduce notation which is used further in the paper. In Section 3, we present the notions of closure operators and closure systems parameterized by systems of isotone Galois connections and show their relationship to parameterized closure structures studied in the past. After presenting examples in Section 4, in Section 5 we investigate general properties of the closure structures and their parameterizations. We give conclusion and final remarks in Section 6.

2. Preliminaries

In the paper, we use the usual notions from the theory of ordered sets and lattices [19,24]. Recall that a partial order $\leq$ on an non-empty set L is a binary relation which is reflexive, antisymmetric, and transitive; the pair $\langle L, \leq \rangle$ is called a partially ordered set. Furthermore, $\langle L, \leq \rangle$ is called a complete lattice and denoted $\mathbf{L} = \langle L, \leq \rangle$ whenever each $K \subseteq L$ has its supremum and infimum in L which are denoted by $\bigvee K$ and $\bigwedge K$, respectively. Each complete lattice $\mathbf{L}$ has its greatest and least elements $1 = \bigvee L = \bigwedge \emptyset$ and $0 = \bigwedge L = \bigvee \emptyset$.

A non-empty set $K \subseteq L$ is called an $\leq$-filter in $\mathbf{L}$ if for every $a, b \in L$ such that $a \leq b$ we have $b \in K$ whenever $a \in K$.

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