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A factorization algorithm for G-algebras and its applications

Viktor Levandovskyy [a], Albert Heinle [b]

[a] Lehrstuhl B für Mathematik, RWTH Aachen University, 52062 Aachen, Germany
[b] David R. Cheriton School of Computer Science, University of Waterloo, ON N2L 3G1, Canada

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ABSTRACT

Factorization of polynomials into irreducible factors is an important problem with numerous applications, both over commutative and over non-commutative rings. It has been recently proved by Bell, Heinle and Levandovskyy that a large class of non-commutative algebras are finite factorization domains (FFD for short). This provides a termination criterion for a factorization algorithms of elements in a vast class of finitely presented $\mathbb{K}$-algebras, which includes the ubiquitous G-algebras, encompassing algebras of common linear partial functional operators.

In this paper, we contribute an algorithm to find all distinct factorizations of a given element in a G-algebra, with minor assumptions on the underlying field, and establish its complexity. Moreover, the property of being an FFD, in combination with the factorization algorithm, enables us to generalize the factorizing Gröbner basis algorithm for G-algebras. This algorithm is useful for various applications, e.g. in analysis of solution spaces of systems of linear partial functional equations with polynomial coefficients. Additionally, it is possible to include inequality constraints for ideals in the input.

The developed algorithms are accompanied by freely available implementations and illustrated by interesting examples.

E-mail addresses: levandov@math.rwth-aachen.de (V. Levandovskyy), aheinle@uwaterloo.ca (A. Heinle).
URL: http://www.math.rwth-aachen.de/~levandov (V. Levandovskyy).

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1. Introduction

Notations: Throughout the paper we denote by $\mathbb{K}$ a field. In the algorithmic part we will assume $\mathbb{K}$ to be a computable field. $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is the set of natural numbers including zero. For a $\mathbb{K}$-algebra R we denote by $U(R)$ the group of invertible elements (units) of R, which is nonabelian in general. For $f \in R$ we denote by $Rf = {}_R\langle f \rangle$ the left ideal, generated by f. If no confusion arises, we also use the notation $\langle f \rangle$. Under $\mathrm{NF}(f, I)$ we mean the left normal form of an element f with respect to a left ideal I. The main focus in this paper lies in so called G-algebras, which are defined as follows.

Definition 1. For $n \in \mathbb{N}$ and $1 \le i < j \le n$ consider the units $c_{ij} \in \mathbb{K}^*$ and polynomials $d_{ij} \in \mathbb{K}[x_1, \dots, x_n]$. Suppose, that there exists a monomial total well-ordering $\prec$ on $\mathbb{K}[x_1, \dots, x_n]$, such that for any $1 \le i < j \le n$ either $d_{ij} = 0$ or the leading monomial of d_{ij} is smaller than $x_i x_j$ with respect to $\prec$. The $\mathbb{K}$-algebra $A := \mathbb{K}\langle x_1, \dots, x_n \mid \{x_j x_i = c_{ij} x_i x_j + d_{ij} : 1 \le i < j \le n\}\rangle$ is called a *G-algebra*, if $\{x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n} : \alpha_i \in \mathbb{N}_0\}$ is a $\mathbb{K}$-basis of A.

G-algebras (Apel, 1988; Levandovskyy, 2005) are also known as algebras of solvable type (Kandri-Rody and Weispfenning, 1990; Li, 2002) and as PBW algebras (Bueso et al., 2001, 2003). G-algebras are Noetherian domains of finite global, Krull and Gel'fand–Kirillov dimensions.

We assume that the reader is familiar with the basic terminology in the area of Gröbner bases, both in the commutative as well as in the non-commutative case. We recommend Buchberger (1997), Bueso et al. (2003), Levandovskyy (2005) as literature on this topic.

Recall, that $r \in R \setminus \{0\}$ is called **irreducible**, if in any factorization $r = ab$ either $a \in U(R)$ or $b \in U(R)$ holds. Otherwise, we call r reducible. Also, r is called **central**, if $\forall t \in R\ rt = tr$ holds.

Definition 2. We say that a domain A is a *finite factorization domain* (FFD, for short), if every nonzero, non-unit element of A has at least one factorization into irreducible elements and there are at most finitely many distinct factorizations into irreducible elements up to multiplication of the irreducible factors by central units in A.

A word of warning: at least since N. Jacobson (Jacobson, 1943), in ring theory (Cohn, 2006) a different definition of a factorization has been used. In the context of Definition 2 two elements are called *associated* if they differ (like in the classical commutative case) by a (central) unit. In this case one writes $a \sim b$. Consider the following relation (called *left similarity* in Bueso et al., 2003) for a, b in a ring R: $a \approx b \Leftrightarrow R/Ra \cong R/Rb$ as left R-modules. Then it can be proved that some noncommutative domains are even domains of unique factorization with respect to $\approx$ (we denote this by $\mathrm{UFD}_{\approx}$). That is, if $a_1 \cdot \ldots \cdot a_s = b_1 \cdot \ldots \cdot b_t$ and a_i, b_j are non-units from a domain R, which cannot be written as a product of two units, then $s = t$ and there exists a permutation σ such that $\forall i\ a_i \approx b_{\sigma(i)}$ holds. In order to realize, how different these two definitions of factorization and associatedness actually are, let us consider the following important cases.

(i) A free associative algebra $\mathbb{K}\langle x_1, \dots, x_n \rangle$ is an $\mathrm{UFD}_{\approx}$ (e.g. Cohn, 1973, Theorem 6.3). Let $n = 2$, consider $\mathbb{K}\langle x, y \rangle \ni f = xyx - x$. Then one can show, that $f = x(yx - 1) = (xy - 1)x$ are all factorizations of f up to central units. Moreover, from the main Theorem from Bell et al. (2017) it follows, that $\mathbb{K}\langle x_1, \dots, x_n \rangle$ is an FFD, but as we can see, it is not $\mathrm{UFD}_{\sim}$ but $\mathrm{UFD}_{\approx}$.

(ii) A noncommutative left and right principal ideal domain is an $\mathrm{UFD}_{\approx}$ (e.g. Bueso et al., 2003). This encompasses the case of Ore extensions $D[x; \sigma, \delta]$ of a division algebra D by a skew derivation (σ, δ) with $\sigma \in \mathrm{Aut}(D)$. Recall the first rational Weyl algebra $B_1 = \mathbb{K}(x)[\partial; \mathrm{id}, \frac{d}{dx}]$, which can be viewed as an Ore localization of the first Weyl algebra $A_1 = \mathbb{K}\langle x, \partial \mid \partial x = x\partial + 1\rangle$ at the set $S = \mathbb{K}[x] \setminus \{0\}$. Then B_1 is an $\mathrm{UFD}_{\approx}$, but it is not a FFD in the sense of Definition 2 because of the classical fact, that

$$\forall (b, c) \in \mathbb{C}^2 \setminus \{(0, 0)\}, \quad \partial^2 = \left(\partial + \frac{b}{bx - c}\right) \cdot \left(\partial - \frac{b}{bx - c}\right),$$

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