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Dancing samba with Ramanujan partition congruences

Ralf Hemmecke

Research Institute for Symbolic Computation, Johannes Kepler University, Altenbergerstraße 69, A-4040 Linz, Austria

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ABSTRACT

The article presents an algorithm to compute a $C[t]$-module basis G for a given subalgebra A over a polynomial ring $R = C[x]$ with a Euclidean domain C as the domain of coefficients and t a given element of A. The reduction modulo G allows a subalgebra membership test. The algorithm also works for more general rings R, in particular for a ring $R \subset C((q))$ with the property that $f \in R$ is zero if and only if the order of f is positive. As an application, we algorithmically derive an explicit identity (in terms of quotients of Dedekind η-functions and Klein's j-invariant) that shows that $p(11n+6)$ is divisible by 11 for every natural number n where $p(n)$ denotes the number of partitions of n.

1. Introduction

Ramanujan (1921) discovered that

$$p(5n+4) \equiv 0 \pmod 5 \tag{1}$$

$$p(7n+5) \equiv 0 \pmod 7 \tag{2}$$

$$p(11n+6) \equiv 0 \pmod{11} \tag{3}$$

for all natural numbers $n \in \mathbb{N}$ where $p(n)$ denotes the number of partitions of n. In Ramanujan (1919) he lists the following identities from which (1) and (2) can be concluded.

E-mail address: hemmecke@risc.jku.at.

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$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\prod_{k=1}^{\infty}\frac{(1-q^{5k})^5}{(1-q^k)^6} \tag{4}$$

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\prod_{k=1}^{\infty}\frac{(1-q^{7k})^3}{(1-q^k)^4} + 49q\prod_{k=1}^{\infty}\frac{(1-q^{7k})^7}{(1-q^k)^8} \tag{5}$$

A similar "simple" identity for (3) is not known, although Lehner (1943) gave an identity in terms of ad hoc constructed series A and C.

$$q\prod_{k=1}^{\infty}(1-q^{11k})\sum_{n=0}^{\infty} p(11n+6)q^n = 11(11AC^2 - 11^2C + 2AC - 32C - 2)$$

Radu (2015) developed an algorithmic machinery based on modular functions. He first computed generators $M_1, \ldots, M_7$ of the monoid of all quotients of Dedekind η-functions of level 22 that only have poles at infinity. For more details see Radu (2015). In terms of q-series, these generators are as follows.

$$M_1 = q^{-5}\prod_{k=1}^{\infty}\frac{(1-q^k)^7(1-q^{11k})^3}{(1-q^{2k})^3(1-q^{22k})^7}$$

$$M_2 = q^{-5}\prod_{k=1}^{\infty}\frac{(1-q^{2k})^8(1-q^{11k})^4}{(1-q^k)^4(1-q^{22k})^8}$$

$$M_3 = q^{-6}\prod_{k=1}^{\infty}\frac{(1-q^{2k})^6(1-q^{11k})^6}{(1-q^k)^2(1-q^{22k})^{10}}$$

$$M_4 = q^{-5}\prod_{k=1}^{\infty}\frac{(1-q^{2k})(1-q^{11k})^{11}}{(1-q^k)(1-q^{22k})^{11}}$$

$$M_5 = q^{-7}\prod_{k=1}^{\infty}\frac{(1-q^{2k})^4(1-q^{11k})^8}{(1-q^{22k})^{12}}$$

$$M_6 = q^{-8}\prod_{k=1}^{\infty}\frac{(1-q^k)^2(1-q^{2k})^2(1-q^{11k})^{10}}{(1-q^{22k})^{14}}$$

$$M_7 = q^{-9}\prod_{k=1}^{\infty}\frac{(1-q^k)^4(1-q^{11k})^{12}}{(1-q^{22k})^{16}}$$

Note that each of these series lives in $\mathbb{Z}((q))$. By his algorithms AB and MW, Radu then computes a relation

$$\begin{aligned} F &= 11(98t^4 + 1263t^3 + 2877t^2 + 1019t - 1997) \\ &\quad + 11z_1(17t^3 + 490t^2 + 54t - 871) + 11z_2(t^3 + 251t^2 + 488t - 614) \end{aligned} \tag{6}$$

where F is defined as on top of p. 30 of Radu (2015), i.e.,

$$F = q^{-14}\prod_{k=1}^{\infty}\frac{(1-q^k)^{10}(1-q^{2k})^2(1-q^{11k})^{11}}{(1-q^{22k})^{22}}\sum_{n=0}^{\infty} p(11n+6)q^n \tag{7}$$

and t, z_1, z_2 are given by