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Many disjoint edges in topological graphs

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ABSTRACT

A *monotone cylindrical* graph is a topological graph drawn on an open cylinder with an infinite vertical axis satisfying the condition that every vertical line intersects every edge at most once. It is called *simple* if any pair of its edges have at most one point in common: an endpoint or a point at which they properly cross. We say that two edges are *disjoint* if they do not intersect. We show that every simple complete monotone cylindrical graph on n vertices contains $\Omega(n^{1-\epsilon})$ pairwise disjoint edges for any $\epsilon > 0$. As a consequence, we show that every simple complete topological graph (drawn in the plane) with n vertices contains $\Omega(n^{\frac{1}{2}-\epsilon})$ pairwise disjoint edges for any $\epsilon > 0$. This improves the previous lower bound of $\Omega(n^{\frac{1}{3}})$ by Suk which was reproved by Fulek and Ruiz-Vargas. We remark that our proof implies a polynomial time algorithm for finding this set of pairwise disjoint edges.

1. Introduction

A *topological graph* is a graph drawn on the plane so that its vertices are represented by points and its edges are represented by Jordan arcs connecting the respective endpoints. Moreover, in topological graphs we do not allow overlapping edges nor edges passing through a vertex. A *topological graph* is *simple* if every pair of its edges meet at most once, either in a common vertex or at a proper crossing. We use the words “vertex” and “edge” in both contexts, when referring to the elements of an abstract graph and also when referring to their corresponding drawings. A graph is *complete* if there is an edge between every pair of vertices. We say that two edges are *disjoint* if they do not intersect. Throughout this note n denotes the number of vertices in a graph.

By applying a theorem of Erdős and Hajnal [5], every complete n-vertex simple topological graph contains $e^{\Omega(\sqrt{\log n})}$ edges that are either pairwise disjoint or pairwise crossing. However, it was thought [15] that this bound is far from optimal. Fox and Pach [9] showed that there exists a constant $\delta > 0$ such that every complete n-vertex simple topological graph contains $\Omega(n^{\delta})$ pairwise crossing edges. In 2003, Pach, Solymosi, and Tóth [15] showed that every complete n-vertex simple topological graph has at least $\Omega(\log^{1/8} n)$ pairwise disjoint edges. This lower bound was later improved by Pach and Tóth [16] to $\Omega(\log n/\log\log n)$. Fox and Sudakov [10] improved this to $\Omega(\log^{1+\epsilon} n)$, where ϵ is a very small constant. Furthermore, the previous two bounds hold for dense simple topological graphs. Pach and Tóth conjectured (see Problem 5, Chapter 9.5 in [3]) that there exists a constant $\delta > 0$ such that every complete n-vertex simple topological graph has at least $\Omega(n^{\delta})$ pairwise disjoint edges. Using the existence of a perfect matching with a low stabbing number for set systems with polynomially bounded dual shattered function [4], Suk [17] settled this conjecture by showing that there are always

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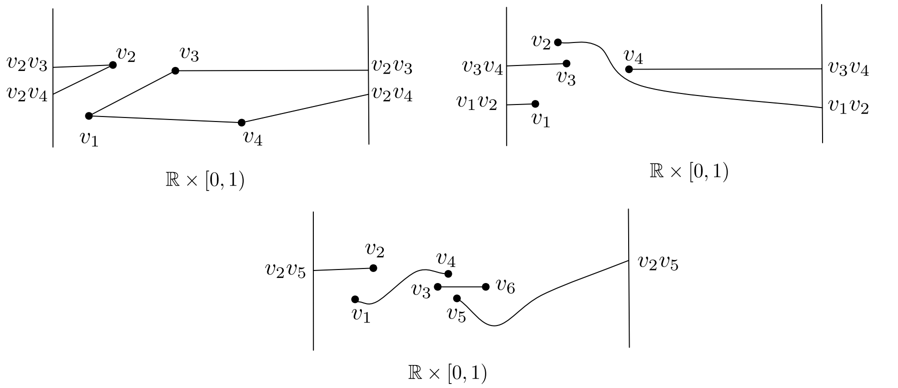

Fig. 1. Three different plane representations of cylindrical graphs.

at least $\Omega(n^{1/3})$ pairwise disjoint edges. This was later reproved by Fulek and Ruiz-Vargas [12] using completely different techniques. Our main result improves the lower bound to $\Omega(n^{1/2-\epsilon})$, for any fixed $\epsilon > 0$.

Theorem 1. *A complete simple topological graph on n vertices, which is drawn on the plane, contains $\Omega(n^{\frac{1}{2}-\epsilon})$ pairwise disjoint edges.*

To the best of our knowledge, no sub-linear upper bound is known for this problem.

Algorithmic aspects Theorem 1 gives a lower bound on the size of a largest independent set in the intersection graph of edges in a complete simple topological graph. Besides the fact that computing the maximum number of pairwise disjoint elements in an arrangement of geometric objects is an old problem in computational geometry, this line of research is also motivated by applications, e.g., in frequency assignment [6], computational cartography [1], and VLSI design [13]. Determining the size of a largest independent set is NP-hard already for intersection graphs of sets of segments in the plane lying in two directions [14], disks [7] and rectangles [2]. For most known cases, efficient algorithms searching for a large independent set in intersection graphs of geometric objects can only approximate the maximum. It is for this reason that a lot of effort has been devoted to finding such approximations (see [8] for more references).

Our proof of Theorem 1 yields an efficient $O(n^3)$ algorithm for finding $\Omega(n^{\frac{1}{2}-\epsilon})$ pairwise disjoint edges in a complete simple topological graph.

In Section 2, we introduce cylindrical graphs and state the necessary results showing that in order to prove Theorem 1 it is enough to show that we can always find $\Omega(n^{1-\epsilon})$ pairwise disjoint edges in every complete monotone simple cylindrical graph. The latter is proved in two steps: in Section 3, for flags (see Section 3 for the definition); in Section 4, for all graphs.

2. Cylindrical drawings of graphs

Let $\mathcal{C}$ be the surface of an infinite open cylinder. Formally, $\mathcal{C} = S^1 \times \mathbb{R}$. We may assume that S^1 is the interval $[0, 1]$ after gluing the point 0 to the point 1. Then, we can characterize each point $p \in \mathcal{C}$ by its coordinates: we use p_x to denote its x-coordinate and p_y to denote its y-coordinate with $0 \le p_x < 1$ and $p_y \in \mathbb{R}$. A *cylindrical graph* is a graph drawn on $\mathcal{C}$ so that its vertices are represented by points and its edges are represented by Jordan arcs connecting the respective endpoints (this is similar to topological graphs with the only difference that the latter are drawn in the plane). As in topological graphs, cylindrical graphs do not allow overlapping edges nor edges passing through a vertex. A *cylindrical graph* is *simple* if every pair of its edges meet at most once either in a common vertex or at a proper crossing.

The straight lines of $\mathcal{C}$ with fixed x-coordinate will be called the *vertical lines* of $\mathcal{C}$. We let $l_{x=a}$ denote the vertical line with x-coordinate equal to a. We say that a curve $\gamma \in \mathcal{C}$ is *x-monotone* if every vertical line intersects γ in at most one point. We say that a cylindrical graph is *monotone* if each of its edges is an x-monotone curve and furthermore no pair of vertices have the same x-coordinate. We will assume that the vertices of any cylindrical graph have x-coordinates distinct from zero.

A monotone cylindrical graph G can be easily represented on the plane: we simply cut $\mathcal{C}$ along the line $l_{x=0}$ and embed the resulting surface $[0, 1) \times \mathbb{R}$ in the plane in the natural way. We call this the *plane representation* of $\mathcal{C}$. For a graph G drawn on $\mathcal{C}$ we will say that the *plane representation* of G is the drawing of G given by the plane representation of $\mathcal{C}$, note that some edges of G might be cut into two connected components while some edges will remain intact, but in both cases, the edges will stay x-monotone curves consisting of either one or two connected pieces, see Fig. 1. Throughout this paper we will refer to the plane representation of $\mathcal{C}$ and G rather than to the actual drawings on $\mathcal{C}$. Hence we also refer to left and right, so for example we say a point $p \in \mathcal{C}$ lies to the *left* (and *right*) of a point $q \in \mathcal{C}$ if $p_x < q_x$ ($p_x > q_x$).

It is easy to see that a complete topological graph drawn on the plane with its edges being x-monotone curves always contains $\lfloor \frac{n}{2} \rfloor$ pairwise disjoint edges. In [12] it was conjectured that a similar statement is true for complete monotone