Computational Geometry 62 (2017) 25–33

Contents lists available at ScienceDirect

Computational Geometry: Theory and Applications

www.elsevier.com/locate/comgeo

On the number of unit-area triangles spanned by convex grids in the plane ☆

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ARTICLE INFO

Article history:
Received 18 May 2016
Accepted 22 December 2016
Available online 30 December 2016

Keywords:
Cobinatorial geometry
Convex sets
Repeated Configurations

ABSTRACT

A finite set of real numbers is called convex if the differences between consecutive elements form a strictly increasing sequence. We show that, for any pair of convex sets $A, B \subset \mathbb{R}$, each of size $n^{1/2}$, the convex grid $A \times B$ spans at most $O(n^{37/17} \log^{2/17} n)$ unit-area triangles. Our analysis also applies to more general families of sets A, B, known as sets of Szemerédi–Trotter type.

1. Introduction

The problem considered in this paper is to obtain a sharp upper bound on the maximum number of unit-area triangles spanned by n points in the plane. The problem has been posed by Oppenheim in 1967 (see [6]) and has been studied since then in a series of papers [1,3,5,16,17]. The currently best known upper bound, for an arbitrary set of n distinct points in the plane, is due to the first two authors [17]:

Theorem 1 *(Raz and Sharir [17]). The number of unit-area triangles spanned by n points in the plane is* $O(n^{20/9})$.

Erdős and Purdy [5] showed that a $\sqrt{\log n} \times (n/\sqrt{\log n})$ section of the integer lattice determines $\Omega(n^2 \log\log n)$ triangles of the same area. This is the best known lower bound.

In this paper we study a variation of the problem, first considered in [18], concerning the number of unit-area triangles spanned by points in a *convex grid*. That is, the input set is of the form $A \times B$, where A and B are convex sets of $n^{1/2}$ real numbers each; a set $X = \{x_1, \ldots, x_n\}$, with $x_1 < x_2 < \cdots < x_n$, of real numbers is called *convex* if

$$x_{i+1} - x_i > x_i - x_{i-1},$$

for every $i = 2, \ldots, n-1$. See [4,7–10,13,20,21,25] for more details and properties of convex sets.

☆ Work on this paper by Orit E. Raz and Micha Sharir was supported by Grant 892/13 from the Israel Science Foundation, and by the Israeli Centers for Research Excellence (I-CORE) program (Center No. 4/11). Work by Micha Sharir was also supported by Grant 2012/229 from the U.S.–Israel Binational Science Foundation, and by the Hermann Minkowski-MINERVA Center for Geometry at Tel Aviv University.

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http://dx.doi.org/10.1016/j.comgeo.2016.12.002
0925-7721/

Convex sets are usually studied within the realm of additive combinatorics, and are considered to be one of the simplest objects in this area. Normally, they are analyzed as subsets of the real line, so it is interesting to see that their additive combinatorial properties can be exploited in a "real", two-dimensional geometric problem to yield improved bounds.

Specifically, we establish the following improvement of Theorem 1 for convex grids.

Theorem 2. *Let $S = A \times B$, where $A, B \subset \mathbb{R}$ are convex sets of size $n^{1/2}$ each. Then the number of unit-area triangles spanned by the points of S is $O(n^{37/17}\log^{2/17} n)$.*

As noted, the problem was first considered in the conference paper [18], where the weaker bound $O(n^{31/14})$ has been obtained. The analysis presented here uses the general high-level approach as in [18], but develops and adapts refined techniques that lead to the improved bound asserted above.[1]

Our result is a somewhat unusual application of the properties of convex sets (and of their generalization, so-called *sets of Szemerédi–Trotter type*, defined and discussed below) in a two-dimensional "purely geometric" context, in contrast with their more standard applications in arithmetic combinatorics (such as in [13,21]). Nevertheless, to obtain the improved bounds, we need to adapt and develop new properties of convex sets (and of sets of Szemerédi–Trotter type) within their standard arithmetic context. These properties are presented in Section 2, and culminate in Lemma 9, a technical "multi-dimensional" property of such sets, which we believe to be of independent interest.

In other words, besides the result that the number of unit-area triangles spanned by a convex grid admits a better upper bound than the one known for the general case, another merit of this paper is that this geometric problem motivates the study of, and actually derives, further properties of convex sets (and of sets of Szemerédi–Trotter type), which hopefully will also find further applications.

The main technical tool used in our analysis is a result of Schoen and Shkredov [21] on difference sets involving convex sets. For two given finite subsets $X, Y \subset \mathbb{R}$, and for any $s \in \mathbb{R}$, denote by $\delta_{X,Y}(s)$ the number of representations of s in the form $x - y$, with $x \in X$, $y \in Y$.

Lemma 3 *(Schoen and Shkredov [21, Lemma 2.6]). Let X, Y be two finite subsets of $\mathbb{R}$, with X convex. Then, for any $\tau \geqslant 1$, we have*

$$\left|\left\{s \in X - Y \mid \delta_{X,Y}(s) \geqslant \tau\right\}\right| = O\left(\frac{|X||Y|^2}{\tau^3}\right).$$

As a matter of fact, our analysis applies to the more general setup of grids $A \times B$, where the sets A, B are of *Szemerédi–Trotter type*. We recall their definition from [22] (with a slightly different notation, and for the special case of the additive group $\mathbb{R}$). We note that sets of Szemerédi–Trotter type have been instrumental in recent works in additive combinatorics, including an improved bound on the sum-product problem [12]; see also [23].

Definition 4 *(Shkredov [22]).* A finite set $X \subset \mathbb{R}$ is said to be of *Szemerédi–Trotter type* (abbreviated as *SzT-type*), with a parameter $\alpha \geqslant 1$, if there exists a parameter $d(X) > 0$ such that

$$\left|\left\{s \in X - Y \mid \delta_{X,Y}(s) \geqslant \tau\right\}\right| \leqslant \frac{d(X)|X||Y|^{\alpha}}{\tau^3}, \tag{1}$$

for every finite set $Y \subset \mathbb{R}$ and every real number $\tau \geqslant 1$.

In view of Lemma 3, any convex set X is of SzT-type, with the parameter $\alpha = 2$ and with $d(X) = O(1)$; see [21] for more details.[2]

We note also that any set X is of SzT-type with parameter $\alpha = 2$ and $d(X) = |X|$. Indeed, it is sufficient to check the condition (1) just for $\tau \leqslant \min\{|X|, |Y|\} \leqslant \sqrt{|X| \cdot |Y|}$. Since the left-hand side of (1) does not exceed $|X||Y|/\tau$, the claim follows. Thus, in what follows we may assume that $d(X) \leqslant |X|$.

Definition 4 is rather general, and applies, with nontrivial bounds for $d(X)$, to several other families of sets, where, for many instances, the condition (1) is obtained using the Szemerédi–Trotter theorem [26] (which is why they are named this way), that we recall next.

Theorem 5 *(Szemerédi and Trotter [26]). (i) The number of incidences between M distinct points and N distinct lines in the plane is $O(M^{2/3}N^{2/3} + M + N)$. (ii) Given M distinct points in the plane and a parameter $k \leqslant M$, the number of lines incident to at least k of the points is $O(M^2/k^3 + M/k)$. Both bounds are tight in the worst case.*

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[1] We remark that the results in the conference paper [18] have been split into two papers, one, by Raz and Sharir [17], covers the general case and one special case (of points lying on three lines), and the present paper, which studies the case of a convex grid, with an improved upper bound.

[2] Note that the factor $d(X)|X|$ in the right-hand side of (1) can be replaced simply by a function of X, $c(X)$, as is done in [22]. We choose to keep the quantity $|X|$ in the notation, though, to have a better comparison between the convex case and the more general case of sets of SzT-type.