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On orthogonally convex drawings of plane graphs [☆]

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ABSTRACT

We investigate the bend-minimization problem with respect to a new drawing style called an *orthogonally convex drawing*, which is an orthogonal drawing with an additional requirement that each inner face is drawn as an *orthogonally convex polygon*. For the class of biconnected plane graphs of maximum degree 3, we present a necessary and sufficient condition for the existence of a no-bend orthogonally convex drawing, which in turn, enables a linear time algorithm to check and construct such a drawing if one exists. We also develop a flow network formulation for bend-minimization in orthogonally convex drawings, yielding a polynomial time solution for the problem. An interesting application of our orthogonally convex drawing technique is to characterize internally triangulated plane graphs that admit floorplans using only orthogonally convex modules subject to given orthogonally convex boundary constraints.

1. Introduction

A *straight line drawing* of a graph is a planar drawing where all edges are drawn as straight lines. It is one of the most fundamental drawing styles and has a long history of study, beginning with the celebrated Fáry's theorem, which states that every planar graph has a straight line drawing [8]. A *convex drawing*, which is another classical drawing style, is a straight line drawing where all faces are drawn as convex polygons. Reflecting the fact that convexity is an important aesthetic criterion in graph drawing, convex drawings have received a lot of research attention over the years. See [20] and Chapters 4 and 5 of [13] for more details.

An *orthogonal drawing* of a plane graph is a planar drawing such that each edge is composed of a sequence of horizontal and vertical line segments with no crossings. Motivated by a variety of applications in VLSI layouts, floor-planning, and database diagram drawings, a considerable amount of works on orthogonal drawings have been carried out in the past. See [7] and Chapters 6–9 of [13].

In contrast to the well-studied convex drawings, very little is known about convexity in orthogonal drawings. One may attempt to regard *rectangular drawings* (a well-studied special case of orthogonal drawings where all faces are drawn as rectangles) as the "convex" versions of orthogonal drawings, as rectangles are exactly "convex" orthogonal polygons. However, using only rectangles seems to be too restricted. In fact, only rather specialized subclasses of plane graphs enjoy such drawings. If a graph is not internally triconnected, then it does not have a rectangular drawing [19].

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In this paper we introduce a new type of orthogonal drawings called *orthogonally convex drawings*, requiring that each inner face be an *orthogonally convex polygon*. A polygon is *orthogonally convex* if for any horizontal or vertical line, if two points on the line are inside of the polygon, then the entire line segment between these two points is also inside of the polygon. Note that if we consider standard convexity instead of orthogonal convexity in the setting of no-bend orthogonal drawings, *inner rectangular drawings* arise, which were studied in [12].

Our work on convexity in orthogonal drawings falls into the line of research of extending or refining the model of orthogonal drawings to better comply with various requirements. To allow graphs of degree more than 4 to be drawn, the *quasi-orthogonal drawing* model was invented in [10]. To improve the readability and aesthetic feel, a new model called a *slanted orthogonal drawing* was introduced in [4]. In this model, a 90° bend is replaced by two 135° bends in order to smoothen the edges. A *smooth orthogonal drawing* adopts another approach to smoothing the bends by replacing bends with circular arcs [3], and this drawing style has found applications in drawing syntax diagrams [2].

Apart from the aesthetic concern, the study of our new drawing style is also motivated by an attempt to learn more about the geometric aspect of orthogonal drawings. In some applications like floor-planning and contact graph representations, one may need to design graph drawing algorithms that respect various shape or area constraints associated with faces. The issue has received some attention in the study of *rectangular duals* and *rectilinear duals* (the dual settings of rectangular drawings and orthogonal drawings) [1,11,17,21]. In this paper we will also study orthogonal convexity in the dual setting.

Bend-minimization is a classical optimization problem in graph drawing, whose goal is to search for a drawing using the minimum number of *bends*. We will study this optimization problem in the setting of orthogonally convex drawings. The bend-minimization problem of orthogonal drawings is NP-hard in the most general setting, i.e., for planar graphs of maximum degree 4 [9]. Subclasses of graphs with bend-minimization of orthogonal drawings solvable in polynomial time include planar graphs of maximum degree 3, series-parallel graphs, and graphs with fixed embeddings [6,18], etc.

The approaches reported in the literature for designing orthogonal drawing algorithms can be roughly divided into two categories, one uses flow or matching to model the problem (e.g., [5,6,18]), while the other tackles the problem in a more graph-theoretic way by taking advantage of structural properties of graphs (e.g., [14–16]). The former usually solves a more general problem, but often requires higher time complexity. On the contrary, algorithms in the latter focus on specific kinds of graphs, resulting in linear time complexity in many cases. As we shall see in our subsequent discussion, the technique used in this paper involves a mixture of the above two types of strategies.

In this paper we gives a comprehensive study of orthogonal convexity in orthogonal drawings. The main contributions of our work are summarized in the following:

1. The drawing style of *orthogonally convex drawings* is proposed. A necessary and sufficient condition, along with a linear time testing algorithm, is presented for biconnected plane 3-graphs (i.e., biconnected plane graphs of maximum degree 3) to admit a no-bend orthogonally convex drawing.
2. We present an alternative characterization for no-bend orthogonally convex drawings of biconnected plane 3-graphs. Based on this characterization, we prove the following results regarding the trade-off between orthogonal convexity and the number of bends:
   - For any triconnected plane 3-graph, the minimum number of bends required is the same regardless of whether the drawing is orthogonally convex or simply orthogonal.
   - For any subdivision of a triconnected plane 3-graph, its orthogonally convex drawing requires at most one more bend than its orthogonal counterpart.
3. Also based on the above alternative characterization, an algorithm based on min-cost flow, running in $O(n^{1.5}\log^3 n)$ time, is devised for the bend-minimization problem of biconnected plane 3-graphs.
4. Lastly, we apply our analysis of orthogonally convex drawings to characterizing internally triangulated graphs that admit Q-floorplans, which are rectilinear duals using only orthogonally convex polygons such that the outer boundary is an orthogonally convex polygon combinatorially equivalent to a given orthogonally convex polygon Q.

2. Preliminaries

Given a graph $G=(V,E)$, we write $\Delta(G)$ to denote the maximum degree of G. We write $V(P)$ and $E(P)$ to denote the sets of vertices and edges G, respectively. Graph G is called a d-graph if $\Delta(G) \le d$. A graph is *planar* if it can be drawn on a plane without edge crossings. A *plane graph* is a planar graph with a fixed combinatorial embedding and a designated outer face F_O. A graph is *simple* if it has no self-loops or multi-edges. A *multi-graph* is a graph where self-loops are disallowed while multi-edges are allowed.

Unless stated otherwise, paths and cycles are assumed to be *simple* throughout this paper, in the sense that there are no repeated vertices. From Section 4 to Section 7, graphs under the name G are assumed to be biconnected, $\triangle(G) \le 3$, and may have multi-edges.

A drawing of a planar graph divides the plane into a set of connected regions, called *faces*. A *contour* of a face F is the cycle formed by vertices and edges along the boundary of F. Such a cycle is also called a *facial cycle*. The contour of the outer face F_O is denoted as C_O. If G is biconnected, all facial cycles are simple cycles.

We adopt some notations used in [15,16]. A cycle C divides a plane graph G into two regions. The one that is inside (resp., outside) cycle C is called the *interior region* (resp., *outer region*) of C. We use $G(C)$ to denote the subgraph of G that