ARTICLE IN PRESS

Discrete Applied Mathematics ■ (■■■■) ■■■–■■■

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On the signed Roman k-domination: Complexity and thin torus graphs

Zehui Shao [a,b], Sandi Klavžar [c,d,e,*], Zepeng Li [f], Pu Wu [b], Jin Xu [f]

[a] Research Institute of Intelligence Software, Guangzhou University, Guangzhou 510006, China
[b] School of Information Science and Engineering, Chengdu University, Chengdu, 610106, China
[c] Faculty of Mathematics and Physics, University of Ljubljana, Slovenia
[d] Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia
[e] Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia
[f] School of Electronic Engineering and Computer Science, Peking University, Beijing, 100871, China

ARTICLE INFO

Article history:
Received 3 May 2017
Received in revised form 21 July 2017
Accepted 24 July 2017
Available online xxxx

Keywords:
Roman domination
Signed roman 2-domination
Computational complexity
Torus graphs
Discharging

ABSTRACT

A signed Roman k-dominating function on a graph $G = (V(G), E(G))$ is a function $f : V(G) \to \{-1, 1, 2\}$ such that (i) every vertex u with $f(u) = -1$ is adjacent to at least one vertex v with $f(v) = 2$ and (ii) $\sum_{x \in N[w]} f(x) \geq k$ holds for any vertex w. The weight of f is $\sum_{u \in V(G)} f(u)$, the minimum weight of a signed Roman k-dominating function is the signed Roman k-domination number $\gamma_{sR}^k(G)$ of G. It is proved that determining the signed Roman k-domination number of a graph is NP-complete for $k \in \{1, 2\}$. Using a discharging method, the values $\gamma_{sR}^2(C_3 \,\square\, C_n)$ and $\gamma_{sR}^2(C_4 \,\square\, C_n)$ are determined for all n.

1. Introduction

A motivation for the recently introduced signed Roman k-domination is that it combines the properties of the Roman domination [2,6,16] and the signed domination [8,19].

A *Roman dominating function* (RDF) on a graph G is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight $\omega(f)$ of an RDF f is the value $\omega(f) = \sum_{u \in V(G)} f(u)$. The minimum weight of a Roman dominating function on a graph G is called the *Roman domination number* of G.

The recent concept of signed Roman k-domination is defined as follows. Let $G = (V(G), E(G))$ be a graph and k a positive integer. Then a function $f : V(G) \to \{-1, 1, 2\}$ is a *signed Roman k-dominating function* (SRkDF) if (i) every vertex u for which $f(u) = -1$ is adjacent to at least one vertex v for which $f(v) = 2$ and (ii) $\sum_{x \in N[w]} f(x) \geq k$ holds for any vertex w of G, where $N[w] = \{x : wx \in E(G)\} \cup \{w\}$ denotes the *closed neighborhood* of w. The *weight* of f is the value $\sum_{u \in V(G)} f(u)$, and the minimum weight of a signed Roman k-dominating function is the *signed Roman k-domination number* $\gamma_{sR}^k(G)$ of G. Let f be a SR1DF and $S \subseteq V(G)$, we denote $f(S) = \sum_{v \in S} f(v)$.

For an RDF or SRkDF f of G, let $V_i = \{x : f(x) = i\}$. Then for an RDF f of G, (V_0, V_1, V_2) is the ordered partition of $V(G)$ induced by f such that $V_i = \{x : f(x) = i\}$ for $i = 0, 1, 2$; and for an SR1DF f of G, (V_{-1}, V_1, V_2) is the ordered partition of $V(G)$ induced by f such that $V_i = \{x : f(x) = i\}$ for $i = -1, 1, 2$.

* Corresponding author at: Faculty of Mathematics and Physics, University of Ljubljana, Slovenia.
E-mail addresses: zshao@cdu.edu.cn (Z. Shao), sandi.klavzar@fmf.uni-lj.si (S. Klavžar).

http://dx.doi.org/10.1016/j.dam.2017.07.026
0166-218X/

Please cite this article in press as: Z. Shao, et al., On the signed Roman k-domination: Complexity and thin torus graphs, Discrete Applied Mathematics (2017), http://dx.doi.org/10.1016/j.dam.2017.07.026.

The signed Roman k-domination was introduced by Henning and Volkmann in [12], generalizing the case $k = 1$ studied earlier in [1]. The paper [12] gives different bounds and exact results on $\gamma^k_{sR}(G)$. Among other results, $\gamma^2_{sR}(C_n)$ and $\gamma^k_{sR}(K_{p,p})$ $(p \geq k-1)$ are determined. In the subsequent paper [11] (interestingly, published a year earlier!) the same authors improved a lower bound from [12] on γ^k_{sR} for trees and characterized the trees achieving equality. Volkmann [17] further extended the signed Roman k-domination to digraphs, again generalizing the case $k = 1$ that was first studied in [15]. Very recently, signed *total* Roman domination in digraphs and signed Roman *edge* k-domination were introduced and investigated in [18] and [3], respectively.

Clearly, the signed Roman k-domination number is defined only for graphs G with $\delta(G) \geq k/2 - 1$, where $\delta(G)$ is the minimum degree of G. However, as pointed out in [12], it is reasonable to assume that $\delta(G) \geq k - 1$. Since in this paper we restrict our attention to the cases $k = 1$ and $k = 2$, this assumption requires only that graphs considered have no isolated vertices.

The signed Roman k-domination problem is the following:

Signed Roman k-Domination Problem	
Input:	A graph G, and an integer ℓ.
Question:	Is there an SRkDF of G with weight at most ℓ?

Our first main result to be proved in Section 2 is given in the following:

Theorem 1.1. Signed Roman 1-Domination Problem *is NP-complete even when restricted to bipartite and planar graphs.* Signed Roman 2-Domination Problem *is NP-complete even when restricted to planar graphs.*

Recall that the *Cartesian product* $G \,\Box\, H$ of graphs G and H is the graph with the vertex set $V(G) \times V(H)$, where $(g, h)(g', h') \in E(G \,\Box\, H)$ if either $gg' \in E(G)$ and $h = h'$, or $hh' \in E(H)$ and $g = g'$. The Cartesian product operation is commutative and associative, we refer to the book [10] for additional properties of this graph operation. Cartesian products of two cycles are known as *torus graphs* because of their natural embeddings into the torus. The following theorems for thin (meaning that one factor is short) torus graphs will be proved using a discharging method. While discharging is widely applied in graph coloring, cf. [5], as far as we know it has not been used earlier in domination theory.

Theorem 1.2. *If* $n \geq 3$, *then*

$$\gamma^2_{sR}(C_3 \,\Box\, C_n) = \begin{cases} \dfrac{3n}{2}; & n \equiv 0 \pmod 4, \\ \left\lceil \dfrac{3n}{2} \right\rceil + 1; & n \equiv 1, 2, 3 \pmod 4. \end{cases}$$

Theorem 1.3. *If* $n \geq 4$, *then*

$$\gamma^2_{sR}(C_4 \,\Box\, C_n) = \begin{cases} 10; & n = 4, \\ 11; & n = 5, \\ 2n; & n \geq 6. \end{cases}$$

Theorems 1.2 and 1.3 will be proved in Section 3.

Throughout the paper we will use the notation $[n] = \{1, \ldots, n\}$.

2. Proof of Theorem 1.1

Note first that the Signed Roman k-Domination Problem is clearly in NP.

In the rest of the section we are going to give a reduction of the NP-complete Roman domination problem, to Signed Roman 1-Domination Problem and to Signed Roman 2-Domination Problem, where the former problem is defined as follows.

Roman Domination Problem	
Input:	A graph G, and an integer ℓ.
Question:	Is there an RDF of G with weight at most ℓ?

The NP-completeness of the Roman Domination Problem is mentioned in [7]; it remains NP-complete even when restricted to split graphs, bipartite graphs, and planar graphs. All these results follow from a more general result [4, Theorem 1]. In the same paper a review of the NP-hardness of the Roman Domination Problem is also made [4, Section 4.1], and NP-hardness is proved when restricted to line graphs. On the other hand, the Roman domination number can be computed in linear time for several important classes of graphs including interval graphs, cographs [13], and strongly chordal graphs [14].

The reductions are presented in Sections 2.1 and 2.2, respectively.

Please cite this article in press as: Z. Shao, et al., On the signed Roman k-domination: Complexity and thin torus graphs, Discrete Applied Mathematics (2017), http://dx.doi.org/10.1016/j.dam.2017.07.026.