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Incidence coloring of graphs with high maximum average degree

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ABSTRACT

An incidence of an undirected graph G is a pair (v, e) where v is a vertex of G and e an edge of G incident with v. Two incidences (v, e) and (w, f) are adjacent if one of the following holds: (i) $v = w$, (ii) $e = f$ or (iii) $vw = e$ or f. An incidence coloring of G assigns a color to each incidence of G in such a way that adjacent incidences get distinct colors. In 2005, Hosseini Dolama and Sopena (2005) proved that every graph with maximum average degree strictly less than 3 can be incidence colored with $\Delta + 3$ colors. Recently, Bonamy et al. (2014) proved that every graph with maximum degree at least 4 and with maximum average degree strictly less than $\frac{7}{3}$ admits an incidence $(\Delta + 1)$-coloring. In this paper we give bounds for the number of colors needed to color graphs having maximum average degrees bounded by different values between 4 and 6. In particular we prove that every graph with maximum degree at least 7 and with maximum average degree less than 4 admits an incidence $(\Delta + 3)$-coloring. This result implies that every triangle-free planar graph with maximum degree at least 7 is incidence $(\Delta + 3)$-colorable. We also prove that every graph with maximum average degree less than 6 admits an incidence $(\Delta + 7)$-coloring. More generally, we prove that $\Delta + k - 1$ colors are enough when the maximum average degree is less than k and the maximum degree is sufficiently large.

1. Introduction

In the following we only consider simple, non-empty connected graphs. In a graph $G = (V, E)$, an *incidence* is an edge e coupled with one of its two extremities, denoted by (u, uv) or (v, uv). In other words, incidences of G are in natural bijection with edges in the graph G_s obtained from G by subdividing every edge once.

The set of all incidences in G is denoted by $I(G)$, where

$$I(G) = \{(v, e) \in V(G) \times E(G) : \text{edge } e \text{ is incident to } v\}.$$

Two incidences (u, e) and (v, f) are adjacent if one of the following holds :

(i) $u = v$, (ii) $e = f$ and (iii) the edge $uv = e$ or $uv = f$.

A *strong incidence of* a vertex u is an incidence (u, uv) for some v. A *weak incidence of* a vertex u is an incidence (v, uv) for some v. In both cases, a strong or weak incidence of u can be referred to as an *incidence of* u.

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A proper *incidence coloring* of G is a coloring of the incidences in such a way that for every vertex u, a strong incidence of u does not receive the same color as any other incidence of u. It is equivalent to say that adjacent incidences get distinct colors. This corresponds to an edge coloring of G_s such that two incident edges receive different colors, and no edge is incident with two edges of the same color (this is called a *strong edge coloring*). We say that G is *incidence k-colorable* if it can be properly incidence colored using only integers between 1 and k. Note that this definition allows for a non-integer value of k, though $\lfloor k \rfloor$ could be equivalently considered. We denote by $\chi_i(G)$ the *incidence chromatic number* of G, which is the smallest integer k such that G is incidence k-colorable. The notion of incidence coloring was first introduced by Brualdi and Massey [3]. And they posed the Incidence Coloring Conjecture, which states that:

Conjecture 1 (*Brualdi and Massey [3]*)**.** *For every graph G, $\chi_i(G) \leq \Delta(G) + 2$.*

When Brualdi and Massey defined this variant of coloring, they also provided tight bounds for its corresponding chromatic number, as follows.

Theorem 1 (*Brualdi and Massey [3]*)**.** *For every graph G, $\Delta(G) + 1 \leq \chi_i(G) \leq 2\Delta(G)$.*

However, in 1997, by observing that the concept of incidence coloring is a particular case of directed star arboricity introduced by Algor and Alon [1], Guiduli [4] disproved the Incidence Coloring Conjecture showing that Paley graphs have an incidence chromatic number at least $\Delta + \Omega(\log \Delta)$. He also improved the upper bound proposed by Brualdi and Massey in Theorem 1.

Theorem 2 (*Guiduli [4]*)**.** *For every graph G, $\chi_i(G) \leq \Delta(G) + o(\log \Delta(G))$.*

For technical purpose, we use the stronger notion of incidence coloring introduced by Hosseini Dolama, Sopena and Zhu in 2004 [6], defined as follows. An incidence (k, ℓ)-coloring of G is an incidence k-coloring such that for every vertex u, at most ℓ different colors can appear on weak incidences of u. In particular, one can note that an incidence $(k, 1)$-coloring of a graph is actually tantamount to a square coloring of it, that is, a proper coloring of its vertices with the additional property that no vertex can have two neighbors with the same color. Indeed, we can consider the unique color used on the weak incidences of a vertex to be assigned to that vertex, and conversely. Note again that the notion of incidence (k, ℓ)-coloring holds for non-integer values of k and ℓ.

Let $\mathrm{mad}(G) = \max\left\{\frac{2|E(H)|}{|V(H)|},\ H \subseteq G\right\}$ be the maximum average degree of the graph G, where $V(H)$ and $E(H)$ are the sets of vertices and edges of H, respectively. This is a conventional measure of sparseness of arbitrarily graphs (not necessary planar). For more details on this invariant see [7] where properties of the maximum degree are exhibited and where it is proved that maximum average degree may be computed by a polynomial algorithm. Results linking maximum average degree and incidence coloring date back to 2005, where Hosseini Dolama and Sopena [5] started looking for such relationships in the case of graphs with low maximum average degree (*i.e.* not only bounded, but bounded by a small constant). However, earlier theorems have implications on graphs with bounded maximum average degree (*i.e.* bounded by any constant).

Theorem 3 (*Hosseini Dolama, Sopena and Zhu [6]*)**.** *Let $k \in \mathbb{N}$, and G be a k-degenerate graph. Then G is incidence $(\Delta(G) + 2k - 1, k)$-colorable.*

As a corollary, it holds immediately that for every integer k, a graph G with $\mathrm{mad}(G) < k$, being $(k-1)$-degenerate, is $(\Delta(G) + 2k - 3, k - 1)$-colorable. By allowing a lower bound on the maximum degree $\Delta(G)$, we seek to reduce the number of colors necessary for an incidence coloring and we prove the following result.

Theorem 4. *Let $k \in \mathbb{N}$, and G be a graph with maximum degree $\Delta(G)$ and maximum average degree $\mathrm{mad}(G) < k$.*

1. *If $\Delta(G) \geq \frac{k^2}{2} + \frac{3k}{2} - 2$, then G is incidence $(\Delta(G) + k - 1, k - 1)$-colorable.*
2. *For all $\alpha > 0$, if $\Delta(G) \geq \frac{3\alpha+1}{2\alpha}k - 2$, then G is incidence $(\Delta(G) + (1+\alpha)k - 1, (1+\alpha)k - 1)$-colorable.*

When considering small values of k, Theorem 4.1 cannot compete with specific results. Let us now discuss some such specific cases. The following results were proved for small values of maximum average degree.

Theorem 5. 1. *If G is a graph with $\mathrm{mad}(G) < 3$, then $\chi_i(G) \leq \Delta(G) + 3$* [5].
2. *If G is a graph with $\mathrm{mad}(G) < 3$ and $\Delta(G) \geq 5$, then $\chi_i(G) \leq \Delta(G) + 2$* [5].
3. *If G is a graph with $\mathrm{mad}(G) < \frac{22}{9}$, then $\chi_i(G) \leq \Delta(G) + 2$* [5].
4. *If G is a graph with $\Delta(G) \geq 4$ and $\mathrm{mad}(G) < \frac{7}{3}$, then $\chi_i(G) = \Delta(G) + 1$* [2].

Let us recall results concerning the incidence chromatic number of 3-degenerate graphs.

Theorem 6 (*Hosseini Dolama and Sopena [5]*)**.** *Every 3-degenerate graph G admits an incidence $(\Delta(G) + 4, 3)$-coloring. Therefore, $\chi_i(G) \leq \Delta(G) + 4$.*