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Isomorphism between circulants and Cartesian products of cycles

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ABSTRACT

We give the necessary and sufficient conditions for isomorphism between circulants and Cartesian products of cycles. Based on this result, we prove that the problem of determining if a circulant is isomorphic to a Cartesian product of cycles belongs to **P** problems.

1. Introduction

A *circulant* $G_n(a_1, a_2, \ldots, a_k)$ on n vertices with k pairwise distinct jumps $a_1, a_2, \ldots, a_k$ has vertices $i+a_1, i+a_2, \ldots, i+a_k$ (mod n) adjacent to each vertex i, where for $k \geq j \geq 1$ each $a_j < n$ for directed circulant, and each $a_j < \frac{n+1}{2}$ for undirected circulant. A *Cartesian product* $G = C_{n_1} \Box C_{n_2} \Box \cdots \Box C_{n_k}$ of k cycles $C_{n_1}, C_{n_2}, \ldots, C_{n_k}$ is a graph (respectively digraph) such that the vertex set $V(G)$ equals the Cartesian product $V(C_{n_1}) \times V(C_{n_2}) \times \cdots \times V(C_{n_k})$ and there is an edge (respectively arc) in G from vertex $u = (u_1, u_2, \ldots, u_k)$ to vertex $v = (v_1, v_2, \ldots, v_k)$ if and only if there exists $1 \leq r \leq k$ such that there is an edge (respectively arc) (u_r, v_r) in C_{n_r} and $u_i = v_i$ for all $i \neq r$. In this paper we focus on isomorphism between these two graphs. Graph isomorphism has been extensively studied for the circulants (e.g., [1,5,6]). Muzychuk in [6] proved that isomorphism problem for the circulants belongs to **P** problems. Graph isomorphism was also extensively studied for the Cartesian product of cycles. Aurenhammer et al. proved that isomorphism problem for the Cartesian product of graphs also belongs to **P** problems [2]. Both results are significant since graph isomorphism in general belongs to **NP** class of problems, and yet it is not proved to be either in **P** or **NP**-complete. In this paper we extend the necessary and sufficient conditions for isomorphism between the circulants with two jumps and Cartesian products of two cycles [5] to the arbitrary circulant and Cartesian product of cycles (directed and undirected). Subsequently, we prove that the isomorphism between the circulants and Cartesian products of cycles belongs to **P** problems. The rest of this paper is organized as follows. In Section 2, we present our main results in Theorem 2.4 and Corollary 2.6. In Section 3, we identify the smallest Cartesian product of cycles that is isomorphic to some circulant when the number of jumps is fixed.

2. Main results

We use the following known notation: if G is isomorphic to H then we write $G \simeq H$, else we write $G \not\simeq H$. Before presenting our main result in Theorem 2.4, we need a few preliminary results that will be useful in its proof. First, the connectivity of circulants will be useful due to Boesch and Tindell [4].

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Theorem 2.1 ([4]). *Circulant $G_n(a_1, a_2, \ldots, a_k)$ is connected if and only if $\gcd(n, a_1, a_2, \ldots, a_k) = 1$.* □

Second, isomorphism between circulant $G_n(a_1, a_2)$ and Cartesian product $C_{n_1} \Box C_{n_2}$ will be useful that has been recently published [5].

Theorem 2.2 ([5]). *Circulant $G_n(a_1, a_2)$ is isomorphic to Cartesian product $C_{n_1} \Box C_{n_2}$ if and only if the following conditions hold:*

(1) $n = n_1 n_2$,
(2) $n_1 = \gcd(n, a_j)$ and $n_2 = \gcd(n, a_{3-j})$ where $j = 1$ or $j = 2$,
(3) $\gcd(n_1, n_2) = 1$. □

Theorems 2.1–2.2 apply to directed and undirected graphs. Similarly, our next results are generic in the sense that they apply to directed as well as undirected graphs. In particular, Theorem 2.1 can be extended to disconnected circulants as follows:

Lemma 2.3. *Circulant $G_n(a_1, a_2, \ldots, a_k)$ consists of r connected components if and only if $\gcd(n, a_1, a_2, \ldots, a_k) = r$.*

Proof. If $r = 1$ then by Theorem 2.1 $G = G_n(a_1, a_2, \ldots, a_k)$ consists of a single connected component if and only if $\gcd(n, a_1, a_2, \ldots, a_k) = r = 1$, and we are done. Otherwise, G is a disconnected circulant. Let in this case G^i be an ith connected component in G. Let q be the smallest number such that a connected component G^i contains vertices 1 and $1 + q$ of G. By isomorphism $i \to i + 1$ of G, each G^i is a connected circulant of form $G^i = G^i_{n/q}(a_1/q, a_2/q, \ldots, a_k/q)$. So, $q = r$. Therefore, by Theorem 2.1

$$\gcd\left(\frac{n}{r}, \frac{a_1}{r}, \frac{a_2}{r}, \ldots, \frac{a_k}{r}\right) = 1,$$

which means that

$$r = \gcd\left(\frac{n}{r}, \frac{a_1}{r}, \frac{a_2}{r}, \ldots, \frac{a_k}{r}\right) r = \gcd(n, a_1, a_2, \ldots, a_k). \quad \Box$$

We can now prove our main result. The proof is based on the induction on the number of jumps in circulant or conversely on the number of cycles in Cartesian product of cycles. It is also based on the concept of *arc/edge-symmetric* subgraph of a given graph. To define what arc/edge-symmetry in graph means, we need a few additional definitions derived from [4]. An *automorphism* is an isomorphism of a graph onto itself. We say that two vertices are *similar* if there is an automorphism that maps one of them onto the other. If there is an automorphism such that the image of vertices x, y of arc/edge (x, y) is the vertices x', y' of another arc/edge (x', y') then we say that arcs/edges (x, y) and (x', y') are *similar*. A subgraph of a graph is *arc/edge-symmetric* if its arcs/edges are pairwise similar. For the purpose of the proof of the following theorem, we also allow a cycle of length two for undirected simple graphs (i.e., edge (v_i, v_j) represents a cycle $v_i v_j v_i$). Hence, any 1-factor in undirected graph G we also consider to be a 2-factor of G with undirected cycles of length 2.

Theorem 2.4. *Circulant $G_n(a_1, a_2, \ldots, a_k)$ is isomorphic to Cartesian product $C_{n_1} \Box C_{n_2} \Box \cdots \Box C_{n_k}$ for $k \geq 2$ if and only if the following conditions hold:*

(1) $n = \prod_{i=1}^{k} n_i$
(2) $\frac{n}{n_i} = \gcd(n, a'_i)$ for $k \geq i \geq 1$ and for some permutation $(a'_1, a'_2, \ldots, a'_k)$ of $(a_1, a_2, \ldots, a_k)$
(3) $\gcd(n_i, n_j) = 1$ for $k \geq i \geq 1$, $k \geq j \geq 1$, and $i \neq j$.

Proof. We proceed by induction on k. If $k = 2$ then conditions (1)–(3) are necessary and sufficient based on Theorem 2.2. Denote $Q_k = C_{n_1} \Box C_{n_2} \Box \cdots \Box C_{n_k}$, and let $R_k = G_{n_1 \cdot n_2 \cdots n_k}(a_1, a_2, \ldots, a_k)$ be a connected circulant. We construct from R_k a disconnected circulant $H_{k+1} = G_{n \cdot n_{k+1}}(a_1 \cdot n_{k+1}, a_2 \cdot n_{k+1}, \ldots, a_k \cdot n_{k+1})$ consisting of n_{k+1} connected circulants by Lemma 2.3. Suppose $Q_k \simeq R_k$ if and only if (1) through (3) are satisfied for $k \geq 2$.

Consider the necessary conditions first. Suppose $Q_{k+1} \simeq R_{k+1}$ for $k \geq 2$. By definitions of Q_{k+1} and R_{k+1}, (1) holds. All arcs/edges induced by any jump a_j of circulant R_{k+1} are by definition similar in R_{k+1}, and they represent 2-factor of identical cycles in R_{k+1}. On the other hand, removing all arcs/edges induced by $C_{n_{k+1}}$ from Q_{k+1} produces disconnected graph Q'_{k+1} with n_{k+1} identical connected components. Every cycle induced by C_{n_x} has exactly one vertex in common with every other cycle induced by C_{n_y} in Q_{k+1}, where $x \neq y$. Furthermore, all arcs/edges induced by any cycle C_{n_i} in Q_{k+1} are by definition similar in Q_{k+1}, and they represent 2-factor of identical cycles in Q_{k+1}. Hence, by arc/edge-symmetry of 2-factor in Q_{k+1} induced by C_{n_i} and by arc/edge-symmetry of 2-factor in R_{k+1} induced by a_j the following isomorphism must hold:

$$R_{k+1} \simeq G_{n_1 \cdot n_2 \cdots n_{k+1}}\left(\frac{nn_{k+1}}{n_1}, \frac{nn_{k+1}}{n_2}, \ldots, \frac{nn_{k+1}}{n_{k+1}}\right).$$

Without loss of generality assume that $R_{k+1} = G_{n_1 \cdot n_2 \cdots n_{k+1}}(\frac{nn_{k+1}}{n_1}, \frac{nn_{k+1}}{n_2}, \ldots, \frac{nn_{k+1}}{n_{k+1}})$. Consequently, the similarity of arcs/edges $(i, i + n) \pmod{nn_{k+1}}$ in R_{k+1} implies that removing arcs/edges induced by jump $a'_{k+1} = \frac{nn_{k+1}}{n_{k+1}} = n$ from R_{k+1} produces disconnected graph H_{k+1} already defined that is a circulant isomorphic to Q'_{k+1}. So, $Q'_{k+1} \simeq H_{k+1}$ implying that $Q_k \simeq R_k$.

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