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Around the Complete Intersection Theorem

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ABSTRACT

In their celebrated paper, Erdos et al. (1961) posed the following question. Let $\mathcal{F}$ be a family of k-element subsets of an n-element set satisfying the condition that $|F \cap G| \geq \ell$ holds for any two members of $\mathcal{F}$ where $\ell \leq k$ are fixed positive integers. What is the maximum size $|\mathcal{F}|$ of such a family? They gave a complete solution for the case $\ell = 1$: the largest family is the one consisting of all k-element subsets containing a fixed element of the underlying set. (One has to suppose $2k \leq n$, otherwise the problem is trivial.) They also proved that the best construction for arbitrary ℓ is the family consisting of all k- element subsets containing a fixed ℓ-element subset, but only for large n's. They also gave an example showing that this statement is not true for small n's.

Later Frankl gave a construction for the general case that he believed to be the best. Frankl, Wilson and Füredi made serious progress towards the proof of this conjecture, but the complete solution was not achieved until 1996 when the surprising news came: Rudolf Ahlswede and Levon Khachatrian have found the proof. They invented the expressive name: Complete Intersection Theorem.

We will show some of the consequences of this deep and important theorem.

1. Introduction

The underlying set will be $[n] = \{1, 2, \ldots, n\}$. The family of all k-element subsets of $[n]$ is denoted by $\binom{[n]}{k}$. Its subfamilies are called *uniform*. A family $\mathcal{F}$ of some subsets of $[n]$ is called *intersecting* if $F \cap G \neq \emptyset$ holds for every pair $F, G \in \mathcal{F}$. It is easy to determine the largest (non-uniform) intersecting family in $2^{[n]}$ since at most one of the complementing pairs can be taken.

Observation 1 (*Erdős, Ko, Rado [3]*). *If $\mathcal{F} \subset 2^{[n]}$ is intersecting then $|\mathcal{F}| \leq 2^{n-1} = 2^n/2$.*

The following trivial construction shows that the bound is sharp.

Construction 1. *Take all subsets of $[n]$ containing the element 1.*

However there are many other construction giving equality in Observation 1. The following one will be interesting for our further investigations.

Construction 2. *If n is odd take all sets of size at least $\frac{n+1}{2}$. If n is even then choose all the sets of size at least $\frac{n}{2} + 1$ and the sets of size $\frac{n}{2}$ not containing the element n.*

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The analogous problem when the intersecting subsets have size exactly $k(k \le \frac{n}{2})$, that is the case of uniform families, is not so trivial.

Theorem 1 (*Erdős, Ko, Rado [3]*). *If $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting where $k \le \frac{n}{2}$ then*

$$|\mathcal{F}| \le \binom{n-1}{k-1}.$$

For a shorter proof see [7]. In this case there is only one extremal construction, mimicking Construction 1.

Construction 3. *Take all subsets of $[n]$ having size k and containing the element 1.*

Already Erdős, Ko and Rado, in their seminal paper considered a more general problem. A family $\mathcal{F} \subset 2^{[n]}$ is *t-intersecting* if $|F \cap G| \ge t$ holds for every pair $F, G \in \mathcal{F}$. They posed a conjecture for the maximal size of non-uniform a t-intersecting family. This conjecture was justified in the following theorem.

Theorem 2 (*Katona [8]*). *If $\mathcal{F} \subset 2^{[n]}$ is t-intersecting then*

$$|\mathcal{F}| \le \begin{cases} \displaystyle\sum_{i=\frac{n+t}{2}}^{n} \binom{n}{i} & \text{if } n+t \text{ is even} \\ \displaystyle\sum_{i=\frac{n+t+1}{2}}^{n} \binom{n}{i} + \binom{n-1}{\frac{n+t-1}{2}} & \text{if } n+t \text{ is odd}. \end{cases}$$

Here the generalization of Construction 1 gives only 2^{n-t}, less than the upper bound in Theorem 2 (if $t > 1$). In order to obtain a sharp construction we have to mimic Construction 2.

Construction 4. *If $n+t$ is even, take all sets of size at least $\frac{n+t}{2}$. If $n+t$ is odd then choose all the sets of size at least $\frac{n+t+1}{2}$ and the sets of size $\frac{n+t-1}{2}$ not containing the element n.*

2. The Complete Intersection Theorem

The problem of t-intersecting families for the uniform case proved to be much more difficult than for the non-uniform case. Erdős, Ko and Rado were able to settle the problem when n is large with respect to k. The dependence of the threshold on t is not interesting here since $1 \le t < k$ can be supposed.

Theorem 3 (*[3]*). *If $\mathcal{F} \subset \binom{[n]}{k}$ is t-intersecting and $n > n(k)$ then*

$$|\mathcal{F}| \le \binom{n-t}{k-t}. \tag{1}$$

They also gave an example for small n when (1) does not hold. Let n be divisible by 4, $k = \frac{n}{2}$ and $t = 2$. The family

$$\mathcal{F} = \left\{F : |F| = \frac{n}{2}, \left|F \cap \left[\frac{n}{2}\right]\right| \ge \left[\frac{n}{4}\right] + 1\right\}$$

is 2-intersecting, since any two members meet in at least two elements in $\left[\frac{n}{2}\right]$. On the other hand the size of this family is more than $\binom{n-2}{n/2-1}$, if $n > 4$. (See e.g. $n = 8$.) They believed that this construction was optimal.

The next step towards the better understanding the situation was when Frankl [4] and Wilson [10] determined the exact value of the threshold $n(k)$ in Theorem 3.

Let us now consider the following generalization of the construction above.

Construction 5. *Choose a non-negative integer parameter i and define the family*

$$\mathcal{A}(n, k, t, i) = \{A : |A| = k, |A \cap [t+2i]| \ge t+i\}. \tag{2}$$

It is easy to see that $\mathcal{A}$ is t-intersecting for each i.

Introduce the following notation:

$$\max_{0 \le i} |\mathcal{A}(n, k, t, i)| = AK(n, k, t).$$