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# New results on word-representable graphs

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ABSTRACT

A graph $G = (V, E)$ is word-representable if there exists a word $w$ over the alphabet $V$ such that letters $x$ and $y$ alternate in $w$ if and only if $(x, y) \in E$ for each $x \neq y$. The set of word-representable graphs generalizes several important and well-studied graph families, such as circle graphs, comparability graphs, 3-colorable graphs, graphs of vertex degree at most 3, etc. By answering an open question from Halldórsson et al. (2011), in the present paper we show that not all graphs of vertex degree at most 4 are word-representable. Combining this result with some previously known facts, we derive that the number of $n$-vertex word-representable graphs is $2^{\frac{n^2}{3}+o(n^2)}$.



## 1. Introduction

Let $G = (V, E)$ be a simple (i.e. undirected, without loops and multiple edges) graph with vertex set $V$ and edge set $E$. We say $G$ is *word-representable* if there exists a word $w$ over the alphabet $V$ such that letters $x$ and $y$ alternate in $w$ if and only if $(x, y) \in E$ for each $x \neq y$.

The notion of word-representable graphs has its roots in the study of the celebrated Perkins semigroup [13,14]. These graphs possess many attractive properties (e.g. a maximum clique in such graphs can be found in polynomial time), and they provide a common generalization of several important graph families, such as circle graphs, comparability graphs, 3-colorable graphs, graphs of vertex degree at most 3 (see [7] for definitions of these families).

Recently, a number of fundamental results on word-representable graphs were obtained in the literature [8–12]. In particular, Halldórsson et al. [9] have shown that a graph is word-representable if and only if it admits a *semi-transitive orientation*. However, our knowledge on these graphs is still very limited and many important questions remain open. For example, how hard is it to decide whether a given graph is word-representable or not? What is the minimum length of a word that represents a given graph? How many word-representable graphs on $n$ vertices are there? Does this family include all graphs of vertex degree at most 4?

The last question was originally asked in [9]. In the present paper we answer this question negatively by exhibiting a graph of vertex degree at most 4 which is not word-representable. This result allows us to obtain an upper bound on the asymptotic growth of the number of $n$-vertex word-representable graphs. Combining this result with a lower bound that follows from some previously known facts, we conclude that the number of $n$-vertex word-representable graphs is $2^{\frac{n^2}{3}+o(n^2)}$.

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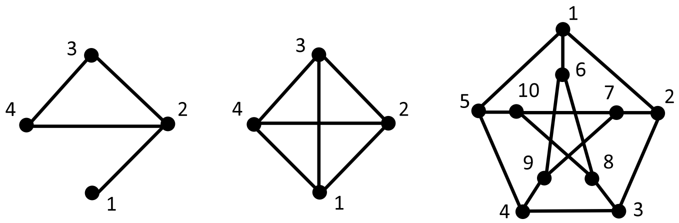


**Fig. 1.** Three word-representable graphs $M$ (left), the complete graph $K_4$ (middle), and the Petersen graph (right).

All preliminary information related to the notion of word-representable graphs can be found in Section 2. In Section 3, we prove our negative result about graphs of degree at most 4 and in Section 4, we derive the asymptotic formula on the number of word-representable graphs.

## 2. Word-representable graphs: definition, examples and related results

Distinct letters $x$ and $y$ *alternate* in a word $w$ if the deletion of all other letters from the word results in either $xyxy\cdots$ or $yxyx\cdots$. A graph $G=(V,E)$ is *word-representable* if there exists a word $w$ over the alphabet $V$ such that letters $x$ and $y$ alternate in $w$ if and only if $(x,y)\in E$ for each $x\neq y$. For example, the graph $M$ in Fig. 1 is word-representable, because the word $w=1213423$ has the right alternating properties, i.e. the only non-alternating pairs in this word are 1, 3 and 1, 4 that correspond to the only non-adjacent pairs of vertices in the graph.

If a graph is word-representable, then there are infinitely many words representing it. For instance, the complete graph $K_4$ in Fig. 1 can be represented by words 1234, 3142, 123412, 12341234, 432143214321, etc. In general, to represent a complete graph on $n$ vertices, one can start with writing up any permutation of length $n$ and adjoining from either side any number of copies of this permutation.

If each letter appears exactly $k$ times in a word representing a graph, the graph is said to be *$k$-word-representable*. It is known [10] that any word-representable graph is $k$-word-representable for some $k$. For example, a 3-representation of the Petersen graph shown in Fig. 1 is

$$1387296(10)7493541283(10)7685(10)194562.$$

It is not difficult to see that a graph is 1-representable if and only if it is complete. Also, with a bit of work one can show that a graph is 2-representable if and only if it is a circle graph, i.e. the intersection graph of chords in a circle. Thus, word-representable graphs generalize both complete graphs and circle graphs. They also generalize two other important graph families, comparability graphs and 3-colorable graphs. This can be shown through the notion of semi-transitive orientation.

A directed graph (digraph) $G=(V,E)$ is *semi-transitive* if it has no directed cycles and for any directed path $v_1v_2\cdots v_k$ with $k\geq 4$ and $v_i\in V$, either $v_1v_k\notin E$ or $v_iv_j\in E$ for all $1\leq i<j\leq k$. In the second case, when $v_1v_k\in E$, we say that $v_1v_k$ is a shortcut. The importance of this notion is due to the following result proved in [9].

**Theorem 1** ([9]). *A graph is word-representable if and only if it admits a semi-transitive orientation.*

From this theorem and the definition of semi-transitivity it follows that all comparability (i.e. transitively orientable) graphs are word-representable. Moreover, the theorem implies two more important corollaries.

**Theorem 2** ([9]). *All 3-colorable graphs are word-representable.*

**Proof.** Partitioning a 3-colorable graph in three independent sets, say I, II and III, and orienting all edges in the graph so that they are oriented from I to II and III, and from II to III, we obtain a semi-transitive orientation. □

**Theorem 3** ([9]). *All graphs of vertex degree at most 3 are word-representable.*

**Proof.** By Brooks' theorem, every connected graph of vertex degree at most 3, except for the complete graph $K_4$, is 3-colorable, and hence word-representable by Theorem 2. Also, as we observed earlier, all complete graphs are word-representable. Therefore, all connected graphs of degree at most 3 and hence all graphs of degree at most 3 are word-representable. □

Whether all graphs of degree at most 4 are word-representable is a natural question following from Theorem 3, which was originally asked in [9]. In the next section, we settle this question negatively.

## 3. A non-representable graph of vertex degree at most 4

The main result of this section is that the graph $A$ represented in Fig. 2 is not word-representable. To prove this, we will show that this graph does not admit a semi-transitive orientation.
Our proof is a case analysis and the following lemma will be used frequently in the proof.