Discrete Applied Mathematics 216 (2017) 162–170

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Minimal dominating sets in interval graphs and trees ☆

Petr A. Golovach [a], Pinar Heggernes [a], Mamadou Moustapha Kanté [b], Dieter Kratsch [c,*], Yngve Villanger [a]

[a] Department of Informatics, University of Bergen, N-5020 Bergen, Norway

[b] Clermont-Université, Université Blaise Pascal, LIMOS, CNRS, Aubiére, France

[c] Université de Lorraine, LITA, Metz, France

ARTICLE INFO

Article history:
Received 15 April 2014
Received in revised form 25 January 2016
Accepted 29 January 2016
Available online 24 February 2016

Keywords:
Graphs
Minimal dominating sets
Graph classes
Enumeration

ABSTRACT

We show that interval graphs on n vertices have at most $3^{n/3} \approx 1.4422^n$ minimal dominating sets, and that these can be enumerated in time $O^*(3^{n/3})$. As there are examples of interval graphs that actually have $3^{n/3}$ minimal dominating sets, our bound is tight. We show that the same upper bound holds also for trees, i.e. trees on n vertices have at most $3^{n/3} \approx 1.4422^n$ minimal dominating sets. The previous best upper bound on the number of minimal dominating sets in trees was 1.4656^n, and there are trees that have 1.4167^n minimal dominating sets. Hence our result narrows this gap. On general graphs there is a larger gap, with 1.7159^n being the best known upper bound, whereas no graph with 1.5705^n or more minimal dominating sets is known.

1. Introduction

Enumerating vertex subsets of a graph satisfying a given property and establishing lower and upper bounds for the maximum number of such vertex subsets in graphs are central tasks in graph algorithms and combinatorics. Branching algorithms are one of the major techniques to design exact exponential algorithms solving NP-hard optimization problems [12]. Such recursive branching algorithms are also a major tool in constructing (exact exponential) enumeration algorithms. Furthermore their running time analysis can be used to obtain upper bounds on the maximum number of enumerated objects in a graph. A classical example is the famous result by Moon and Moser [26], which states that the maximum number of maximal independent sets in a graph on n vertices is $3^{n/3}$. Its proof can be translated into a branching algorithm that enumerates all maximal independent sets of a graph in time $O^*(3^{n/3})$, where the O^*-notation suppresses polynomial factors. The result is tight, as disjoint unions of triangles have exactly $3^{n/3}$ maximal independent sets. Triangle-free graphs, on the other hand, have at most $2^{n/2}$ maximal independent sets [19] and these can be enumerated in time $O^*(2^{n/2})$ [2]. Furthermore this bound is tight, as every 1-regular graph has $2^{n/2}$ maximal independent sets.

Recently there has been extensive research in this direction, both on general graphs and on graph classes, dealing with enumeration algorithms and combinatorial lower and upper bounds of minimal feedback vertex sets, minimal subset feedback vertex sets, minimal separators, and potential maximal cliques [4,9,11,13–15]. Although the above mentioned

☆ This work is supported by the Research Council of Norway (project number 197548/V30), the French National Research Agency (ANR project GraphEn/ANR-15-CE40-0009), and the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013)/ERC Grant Agreement No. 267959.

* Corresponding author. Tel.: +33 387315444.
E-mail addresses: petr.golovach@uib.no (P.A. Golovach), pinar.heggernes@uib.no (P. Heggernes), mamadou.kante@isima.fr (M.M. Kanté), dieter.kratsch@univ-lorraine.fr (D. Kratsch), yngve.villanger@gmail.com (Y. Villanger).

http://dx.doi.org/10.1016/j.dam.2016.01.038
0166-218X/

Table 1
Lower and upper bounds on the maximum number of minimal dominating sets. Note that $15^{n/6} \approx 1.5704^n$ and $3^{n/3} \approx 1.4422^n$.

Graph class	Lower bound	Previous upper bound	This paper
General	$15^{n/6}$	1.7159^n [10]	
Chordal	$3^{n/3}$	1.6181^n [3]	
Split	$3^{n/3}$	$3^{n/3}$ [5]	
Interval	$3^{n/3}$	1.6181^n [3]	$3^{n/3}$
Proper interval	$3^{n/3}$	1.4656^n [3]	$3^{n/3}$
Tree	1.4167^n	1.4656^n [25]	$3^{n/3}$
Co-bipartite	1.3195^n	1.4511^n [5]	
Cograph	$15^{n/6}$	$15^{n/6}$ [3]	

results on maximal independent sets are tight, in general tight bounds are rare and there is often a gap between the best known upper bound and the best known lower bound, i.e., the largest number achieved by a known example. This is in particular the case for minimal dominating sets. Fomin, Grandoni, Pyatkin and Stepanov [10] gave an algorithm with running time $O(1.7159^n)$ for enumerating all minimal dominating sets in an n-vertex graph, thereby showing that the maximum number of minimal dominating sets in such a graph is at most 1.7159^n. However, it is not known whether n-vertex graphs with 1.5705^n or more minimal dominating sets exist [10]. This gap has been narrowed on some well-known graph classes, like chordal graphs [3], trees [25], and cobipartite graphs [5]. Tight bounds have been obtained e.g., on cographs [3] and split graphs [5].

In this paper we study enumeration algorithms and lower and upper bounds for the maximum number of minimal dominating sets in interval graphs and trees. More precisely, we show that every interval graph on n vertices has at most $3^{n/3}$ minimal dominating sets which can be enumerated in time $O^*(3^{n/3})$. The bound is tight as a disjoint union of triangles, which is an interval (even a proper interval) graph, has exactly $3^{n/3}$ minimal dominating sets. Prior to our research reported in this paper, the best known upper bound on the number of minimal dominating sets in interval graphs was 1.6181^n, i.e. the best known upper bound for chordal graphs [3]. Proper interval graphs, which form a subset of interval graphs, have been known to have at most 1.4656^n minimal dominating sets, and hence our result closes the gap on that graph class as well. In addition we improve the upper bound on the number of minimal dominating sets of trees. Krzywkowski [25] has proved an upper bound of 1.4656^n on trees, and he gave a lower bound example with 1.4167^n minimal dominating sets. We show that trees have at most $3^{n/3} \approx 1.4422$ minimal dominating sets which can be enumerated in time $O^*(3^{n/3})$. Our results, together with known results on related graph classes, are summarized in Table 1.

Our study on the number of minimal dominating sets in graphs is motivated from various holds. First of all, domination in graphs is a very well studied subject with many applications in various fields, as can be seen by the number of papers and books published on the subject, see e.g., [18]. Furthermore, the gap between the best known upper and lower bounds on the number of minimal dominating sets in general graphs naturally triggers curiosity about closing the gap on graph classes. Last but not least, enumeration of minimal dominating sets is strongly related to enumeration of minimal transversals of a hypergraph. The latter is a very well studied problem within the field of output polynomial algorithms. In fact, it is a long standing open question of great importance whether there is an output polynomial algorithm to enumerate the minimal transversals of a hypergraph, see e.g., [7,8]. The existence of output polynomial algorithms to enumerate the minimal dominating sets has been studied on graph classes, and several works resolve it positively on some of the graph classes mentioned above [6,7,16,20–23].

2. Preliminaries

Graphs. We work with simple undirected graphs. We denote such a graph by $G = (V, E)$, where V is the set of vertices and E is the set of edges of G, with $n = |V|$ and $m = |E|$. When the vertex set and the edge set of G are not specified, we use $V(G)$ and $E(G)$ to denote these, respectively. The set of neighbors of a vertex $v \in V$ is denoted by $N_G(v)$, and we let $N_G[v] = N_G(v) \cup \{v\}$. For a set $S \subseteq V$, we define analogously $N_G[S] = \bigcup_{v \in S} N_G[v]$ and $N_G(S) = N_G[S] \setminus S$. We will omit the subscript G when there is no ambiguity. A vertex v is *universal* if $N[v] = V$ and *isolated* if $N(v) = \emptyset$. The subgraph of G induced by S is denoted by $G[S]$. We use $G - v$ to denote the graph $G[V \setminus \{v\}]$, and $G - S$ to denote the graph $G[V \setminus S]$. A set $S \subseteq V$ is an *independent set* if $uv \notin E$ for every pair of vertices $u, v \in S$, and S is a *clique* if $uv \in E$ for every pair of vertices $u, v \in S$. A clique (independent set) is *maximal* if no proper superset of it is a clique (independent set).

Domination. A vertex set $D \subseteq V$ is a *dominating set* of G if $N[D] = V$. Every vertex v of a dominating set *dominates* the vertices in $N[v]$. A dominating set D is a *minimal dominating set* if no proper subset of D is a dominating set. If D is a minimal dominating set then for every vertex $v \in D$, there is a vertex $x \in N[v]$ which is dominated only by v. We call such a vertex x a *private neighbor* of v, since x is not adjacent to any vertex in $D \setminus \{v\}$. To avoid confusion we may also call x a private neighbor of v *with respect to D*. Note that a vertex in D might be its own private neighbor. A vertex set $S \subseteq V$ is an *irredundant set* of G if every vertex of S has a private neighbor with respect to S. Observe that every subset of a minimal dominating set is irredundant. We denote the number of minimal dominating sets in a graph G by $\mu(G)$, and $\mu(n)$ denotes