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On recognition of threshold tolerance graphs and their complements

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ABSTRACT

A graph $G = (V, E)$ is a *threshold tolerance* graph if each vertex $v \in V$ can be assigned a weight w_v and a tolerance t_v such that two vertices $x, y \in V$ are adjacent if $w_x + w_y \geq \min(t_x, t_y)$. Currently, the most efficient recognition algorithm for threshold tolerance graphs is the algorithm of Monma, Reed, and Trotter which has an $O(n^4)$ runtime. We give an $O(n^2)$ algorithm for recognizing threshold tolerance and their complements, the co-threshold tolerance (co-TT) graphs, resolving an open question of Golumbic, Weingarten, and Limouzy.

1. Introduction

In [10], Monma, Reed, and Trotter defined a graph $G = (V, E)$ to be a *threshold tolerance graph* if each vertex $v \in V$ can be assigned a weight w_v and a tolerance t_v such that two vertices $x, y \in V$ are adjacent when $w_x + w_y \geq \min(t_x, t_y)$. When the tolerances of the vertices are all the same, we obtain the subclass of *threshold graphs* [4]. Their complements, the *co-threshold tolerance graphs* (*co-TT graphs*), have also received attention as they have an interesting interpretation as a generalization of *interval graphs*.

A graph $G = (V, E)$ is an *interval graph* if and only if each vertex $v \in V$ can be assigned an interval $I_v = [a(v), b(v)]$ on the real line such that two vertices $x, y \in V$ are adjacent exactly when their corresponding intervals intersect, in which case $\mathcal{I} = \{[a(v), b(v)] : v \in V\}$ forms an *interval model* of G. See [3,6,14] for surveys of the properties of this class and its relationship to other graph classes.

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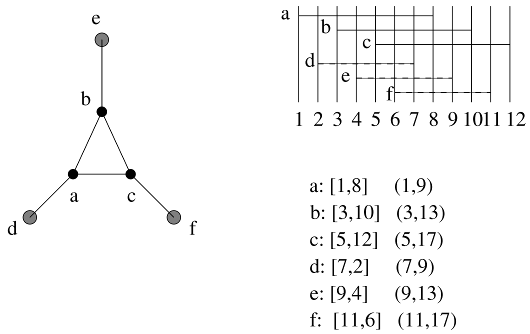

Fig. 1. Vertices that are blue in the model are black and vertices that are red in the model are gray. Though there exist models with as few as one red vertex, there are none where all vertices are blue, hence the graph is not an interval graph. The red vertices are an independent set. A red vertex is adjacent to a blue vertex if its interval is contained in the blue vertex's interval. The pairs on the lower-right illustrates the conversion of interval endpoints to weights and thresholds that represent the complement as a threshold tolerance graph.

To illustrate the relationship of the interval graphs to the co-TT graphs, the definition can be rephrased:

Definition 1. A graph $G = (V, E)$ is an interval graph if and only if there exist functions $a, b : V \mapsto \mathbb{R}$ such that:

- $a(x) \leq b(x)$ for all $x \in V$;
- $xy \in E \Leftrightarrow a(x) \leq b(y) \wedge a(y) \leq b(x)$ for all $x, y \in V$.

By this definition, $[a(x), b(x)]$ is the interval that represents x in the model. Relaxing the requirement that $a(x) \leq b(x)$, gives the class of co-TT graphs:

Definition 2 (*[10]*)**.** A graph $G = (V, E)$ is a co-TT graph if and only if there exist functions $a, b : V \mapsto \mathbb{R}$ such that:

- $xy \in E \Leftrightarrow a(x) \leq b(y) \wedge a(y) \leq b(x)$ for all $x, y \in V$.

It is easy to see that this class is the complement of threshold tolerance graphs by setting $a(x) = w_x$ and $b(x) = t_x - w_x$ for all $x \in V$ gives functions that show that the complement is a co-TT graph, and given the functions $a()$ and $b()$, for a co-TT graph, assigning $w_x = a(x)$ and $t_x = b(x) + w_x$ gives the weights and thresholds that show that its complement is a threshold-tolerance graph.

Following the notation in Golumbic, Weingarten and Limouzy [8], let the *blue–red partition* of V given by a co-TT model be (B, R), where $B = \{x | x \in V$ and $a(x) \leq b(x)\}$ and $R = \{x | x \in V$ and $b(x) < a(x)\}$. Given such a partition, let B be the *blue vertices* and R be the *red vertices*. The *red intervals* are the intervals $[b(x), a(x)]$ corresponding to red vertices and the *blue intervals* are the intervals $[a(x), b(x)]$ corresponding to blue vertices. Collectively, these intervals, together with their coloring, are a *co-TT model*.

The following is easily verified using Definition 2 (see Fig. 1):

Lemma 1 (*[8]*)**.** *Given a co-TT model of a co-TT graph G, let (B, R) be its blue–red partition. Then:*

- *If $\{x, y\} \subseteq B$, then $xy \in E \Leftrightarrow [a(x), b(x)]$ and $[a(y), b(y)]$ intersect;*
- *If $\{x, y\} \subseteq R$, then $xy \notin E$;*
- *If $x \in B$ and $y \in R$, then $xy \in E \Leftrightarrow [b(y), a(y)]$ is contained in $[a(x), b(x)]$.*

It follows that the red vertices are an independent set. Fig. 1 gives an example.

A *chord* on a cycle C in a graph is an edge not on the cycle but whose endpoints are on the cycle. A graph is *chordal* if every cycle on four or more vertices has a chord, see, for example [6]. A chord xy in an even cycle C is *odd* when the distance in C between x and y is odd. A graph is *strongly chordal* if it is chordal and every even-length cycle of size at least six has an odd chord [5].

The following illustrates an interesting relationship between the chordal graphs, the strongly chordal graphs, the co-TT graphs and the interval graphs. A graph is chordal if and only if there is a *perfect elimination ordering*, which is an ordering $(v_1, v_2, \ldots, v_n)$ of its vertices such that for every vertex v_i, v_i and its neighbors to the right form a complete subgraph. It is easily seen that ordering the vertices of an interval or co-TT graph according to left-to-right order of $b()$ in an interval or co-TT model gives a perfect elimination ordering.

A graph is strongly chordal if and only if it has a *simple elimination ordering*, which is an ordering $(v_1, v_2, \ldots, v_n)$ such that for each v_i, the neighbors of v_i in $G[v_i, v_{i+1}, \ldots, v_n]$ are ordered by closed neighborhood containment. That is, if v_j, v_k are two such neighbors, one of $N[v_j]$ and $N[v_k]$ is a subset of the other in $G[v_i, v_{i+1}, \ldots, v_n]$. A simple elimination is a special case of a perfect elimination ordering. Another characterization is that a graph is strongly chordal if and only if it has a *strong elimination ordering*, $(v_1, v_2, \ldots, v_n)$ such that whenever v_i, v_j, v_k are three vertices and $i < j < k$, then $N[v_j] \subseteq N[v_k]$ in $G[v_i, v_{i+1}, \ldots, v_n]$. It is easily seen that ordering the vertices of an interval or co-TT graph according to left-to-right order of $b()$ in an interval or co-TT model also gives a strong elimination ordering, which implies the following theorem.