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Graph editing to a fixed target[☆]

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ABSTRACT

For a fixed graph H, the H-Minor Edit problem takes as input a graph G and an integer k and asks whether G can be modified into H by a total of at most k edge contractions, edge deletions and vertex deletions. Replacing edge contractions by vertex dissolutions yields the H-Topological Minor Edit problem. For each problem we show polynomial-time solvable and NP-complete cases depending on the choice of H. Moreover, when G is AT-free, chordal or planar, we show that H-Minor Edit is polynomial-time solvable for all graphs H.

1. Introduction

Graph editing problems are well studied both within algorithmic and structural graph theory and beyond (e.g. [1,4,22,23]), as they capture numerous graph-theoretic problems with a variety of applications. A graph editing problem takes as input a graph G and an integer k, and the question is whether G can be modified into a graph that belongs to some prescribed graph class $\mathcal{H}$ by using at most k operations of one or more specified types. So far, the most common graph operations that have been considered are vertex deletions, edge deletions and edge additions. Well-known problems obtained in this way are Feedback Vertex Set, Odd Cycle Transversal, Minimum Fill-In, and Cluster Editing. Recently, several papers [12,9,16,17,15] appeared that consider the setting in which the (only) permitted type of operation is that of an *edge contraction*. This operation removes the vertices u and v of the edge uv from the graph and replaces them by a new vertex that is made adjacent to precisely those remaining vertices to which u or v was previously adjacent. So far, the situation in which we allow edge contractions *together with* one or more additional types of graph operations has not been studied. This is the main setting that we consider in our paper.

A natural starting approach is to consider families of graphs $\mathcal{H}$ of cardinality 1, that is, we set $\mathcal{H} = \{H\}$ for some graph H, called the *target* graph from now on, and we assume that H is *fixed*, that is, H is not part of the input. For such families, straightforward polynomial-time algorithms exist if the set of permitted operations may only include edge additions, edge deletions and vertex deletions. If vertex deletions are not permitted, then the input graph G must be of the same order as H yielding a constant-time algorithm as H is assumed to be fixed. If vertex deletions are permitted, then we consider possibly every induced subgraph G' of G that has the same number of vertices as H (say $|V_H| = r$) and verify whether we can modify G' into H by at most $k - r$ edge operations. As H is fixed, such an algorithm takes $O(n^r)$ time (where n denotes the number of vertices of G). However, we show that this approach may no longer be followed in our case, in which we allow both edge contractions and vertex deletions to be applied.

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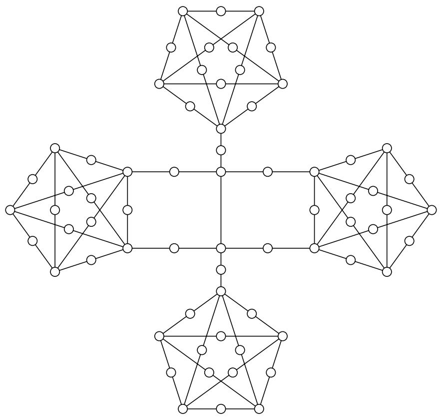

Fig. 1. The smallest graph H^* for which H-Induced Minor is NP-complete [7].

It so happens that setting $\mathcal{H} = \{H\}$ yields graph editing problems that are closely related to problems that ask whether a given graph H appears as a "pattern" within another given graph G so that G can be transformed to H via a sequence of operations *without* setting a bound k on the number of operations allowed. These 'unbounded' problems are ubiquitous in computer science, and below we shortly survey a number of known results on them; those results that we will use in our proofs are stated as lemmas.

We start with some additional terminology. A *vertex dissolution* is the removal of a vertex v with exactly two neighbors u and w, which may not be adjacent to each other, followed by the inclusion of the edge uw. If we can obtain a graph H from a graph G by a sequence that on top of vertex deletions and edge deletions may contain operations of one additional type, namely edge contractions or vertex dissolutions, then G contains H as a *minor* or *topological minor*, respectively. For a *fixed* graph H, that is, H is not part of the input, this leads to the decision problems H-Minor and H-Topological Minor, respectively. Grohe, Kawarabayashi, Marx, and Wollan [13] showed that H-Topological Minor can be solved in cubic time for all graphs H, whereas Robertson and Seymour [25] proved the following seminal result.

Lemma 1 (*[25]*)**.** *H-Minor can be solved in cubic time for all graphs H.*

We say that a containment relation is *induced* if edge deletions are excluded from the permitted graph operations. In the case of minors and topological minors, this leads to the corresponding notions of being an *induced minor* and *induced topological minor*, respectively, with corresponding decision problems H-Induced Minor and H-Induced Topological Minor. In contrast to their non-induced counterparts, the complexity classifications of these two problems have not yet been settled. In fact, the complexity status of H-Induced Minor when H is restricted to be a tree has been open since it was posed at the AMS-IMS-SIAM Joint Summer Research Conference on Graph Minors in 1991. Up until now, only forests on at most seven vertices have been classified [8] (with one forest still outstanding), and no NP-complete cases of forests H are known. The smallest known NP-complete case is the graph H^* on 68 vertices displayed in Fig. 1; this result is due to Fellows, Kratochvíl, Middendorf and Pfeiffer [7].

Lemma 2 (*[7]*)**.** *H^*-Induced Minor is NP-complete.*

Lévêque, Lin, Maffray, and Trotignon [20] gave both polynomial-time solvable and NP-complete cases for H-Induced Topological Minor. In particular they showed the following result, where we denote the complete graph on n vertices by K_n.

Lemma 3 (*[20]*)**.** *K_5-Induced Topological Minor is NP-complete.*

The complexity of H-Induced Topological Minor is still open when H is a complete graph on 4 vertices. Lévêque, Maffray, and Trotignon [21] gave a polynomial-time algorithm for recognizing graphs that neither contain K_4 as an induced topological minor nor a wheel as an induced subgraph. However, they explain that a stronger decomposition theorem (avoiding specific cutsets) is required to resolve the complexity status of K_4-Induced Topological Minor affirmatively.

Before we present our results, we first introduce some extra terminology. Let G be a graph and H a minor of G. Then a sequence of minor operations that modifies G into H is called an *H-minor sequence* or just a minor sequence of G if no confusion is possible. The *length* of an H-minor sequence is the number of its operations. An H-minor sequence is *minimum* if it has minimum length over all H-minor sequences of G. For a fixed graph H, the H-Minor Edit problem is that of testing whether a given graph G has an H-minor sequence of length at most k for some given integer k. Also, for the other containment relations we define such a sequence and corresponding decision problem.

Because any vertex deletion, vertex dissolution and edge contraction reduces a graph by exactly one vertex, any H-induced minor sequence and any H-topological induced minor sequence of a graph G has the same length for any graph H, namely $|V_G| - |V_H|$. Hence, H-Induced Minor Edit and H-Induced Topological Minor Edit are polynomially equivalent to H-Induced Minor and H-Induced Topological Minor, respectively. We therefore do not consider H-Induced Minor Edit