Discrete Applied Mathematics 216 (2017) 273–280

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Vertex coloring of graphs with few obstructions

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ARTICLE INFO

Article history:
Received 7 January 2014
Received in revised form 3 July 2014
Accepted 17 February 2015
Available online 17 March 2015

Keywords:
Vertex coloring
Polynomial-time algorithm
Fixed-parameter tractability

ABSTRACT

We study the vertex coloring problem in classes of graphs defined by finitely many forbidden induced subgraphs. Of our special interest are the classes defined by forbidden induced subgraphs with at most 4 vertices. For all but three classes in this family we show either NP-completeness or polynomial-time solvability of the problem. For the remaining three classes we prove fixed-parameter tractability. Moreover, for two of them we give a 3/2 approximation polynomial algorithm.

1. Introduction

A graph G is *k-colorable* if the vertex set of G can be partitioned into at most k independent sets (color classes). For a fixed value of k, VERTEX-k-COLORABILITY is the problem of deciding whether a given graph G is k-colorable or not. The smallest value of k such that G is k-colorable is called the *chromatic number* of G and is denoted $\chi(G)$.

VERTEX COLORING is the problem of determining $\chi(G)$ and partitioning the vertex set of G into $\chi(G)$ color classes. From a computational point of view, VERTEX COLORING and VERTEX-k-COLORABILITY for any $k \geq 3$ are difficult problems, i.e. they are NP-complete. Moreover, the problems remain NP-complete under substantial restrictions. For instance, VERTEX-3-COLORABILITY (and hence VERTEX COLORING) is NP-complete for triangle-free graphs [21] and for graphs of vertex degree at most four [17], VERTEX-4-COLORABILITY (and hence VERTEX COLORING) is NP-complete for P_7-free graphs [15] and VERTEX-5-COLORABILITY (and hence VERTEX COLORING) is NP-complete for P_6-free graphs [15]. On the other hand, for graphs in some special classes, the problems can be solved in polynomial time. For instance, VERTEX-3-COLORABILITY can be solved for P_6-free graphs [27], VERTEX-4-COLORABILITY can be solved for P_2+P_3-free graphs [12], VERTEX-k-COLORABILITY for any k can be solved for P_5-free graphs [14] and VERTEX COLORING (and hence VERTEX-k-COLORABILITY for any k) can be solved in polynomial time for (triangle, $2P_3$)-free graphs [10] and for perfect graphs.

All classes of graphs mentioned above and all classes we deal with in the present paper possess the property that whenever a graph belongs to a class then every induced subgraph of the graph also belongs to the same class. Such classes are called *hereditary*.

Our goal is systematization of hereditary classes according to the computational complexity of the VERTEX COLORING problem. It is well-known (and not difficult to see) that a class $\mathcal{X}$ of graphs is hereditary if and only if it can be described by a set $\mathcal{Y}$ of forbidden induced subgraphs (obstructions), in which case we write $\mathcal{X} = Free(\mathcal{Y})$. The induced subgraph characterization provides a uniform way to describe hereditary classes and therefore a systematic way to study them. In this paper, we are interested in classes for which the set of obstructions is finite. We call such classes *finitely defined*.

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http://dx.doi.org/10.1016/j.dam.2015.02.015

Of our special interest are the classes defined by forbidden induced subgraphs with at most 4 vertices. To describe the graphs, we use the following notations. By P_n, C_n, K_n and O_n we denote the chordless path, the chordless cycle, the complete graph and the empty (edgeless) graph on n vertices, respectively. Also, $K_n - e$ is the graph obtained from K_n by deleting an edge, and $K_{p,q}$ is the complete bipartite graph with parts of size p and q. By $G_1 + G_2$ we denote the disjoint union of graphs G_1 and G_2, and by kG the disjoint union of k copies of G. In particular, $nK_1 = O_n$, $2K_2 = \overline{C_4}$, $K_2 + 2K_1 = \overline{K_4 - e}$, $C_3 + K_1 = \overline{K_{1,3}}$. The graphs $K_{1,3}$ and $\overline{P_3 + K_1}$ have special names in the literature, they are called *claw* and *paw*, respectively.

Given a graph G, we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. Also, by $L(G)$ we denote the line graph of G, i.e. the graph with vertex set $E(G)$ in which two vertices are adjacent if and only if the respective edges of G share a vertex. It is well-known (see e.g. [13]) that the class of line graphs is hereditary and is defined by 9 forbidden induced subgraphs, one of which is the claw and 8 others contain $K_4 - e$ as an induced subgraph.

The importance of the class of line graphs for our study is due to the fact that VERTEX COLORING restricted to the class of line graphs is equivalent to EDGE COLORING, which is one more important NP-complete problem. From this relationship, we immediately conclude that VERTEX COLORING is NP-complete in the class $Free(K_{1,3})$. More generally, it was shown in [19] that

Theorem 1. *If G is an (not necessarily proper) induced subgraph of P_4 or $P_3 + K_1$, then* VERTEX COLORING *is polynomial-time solvable in the class $Free(G)$. Otherwise, it is NP-complete in $Free(G)$.*

The polynomial-time solvability of VERTEX COLORING in the class $Free(P_4)$ can also be derived from a more general result (Theorem 2 below) proved in [28], since the clique-width of graphs in $Free(P_4)$ is at most 2.

Theorem 2. VERTEX COLORING *is polynomial-time solvable for graphs of bounded clique-width.*

In [9], the family of hereditary classes of graphs defined by forbidden induced subgraphs with at most 4 vertices was completely characterized in terms of bounded or unbounded clique-width. For classes where the clique-width is bounded, we immediately conclude polynomial-time solvability of the VERTEX COLORING problem by Theorem 2. In this paper, we study computational complexity of the problem on classes of unbounded clique-width. According to [9], in the family of classes defined by forbidden induced subgraphs with at most 4 vertices there are seven minimal classes of unbounded clique-width. These are:

$\mathcal{X}^1 = Free(K_{1,3}, C_4, K_4 - e, K_4)$,
$\mathcal{X}^2 = Free(\overline{K_{1,3}}, \overline{C_4}, \overline{K_4 - e}, \overline{K_4})$,
$\mathcal{X}^3 = Free(K_3, C_4)$,
$\mathcal{X}^4 = Free(O_3, \overline{C_4})$,
$\mathcal{X}^5 = Free(K_4, 2K_2)$,
$\mathcal{X}^6 = Free(C_4, 2K_2)$,
$\mathcal{X}^7 = Free(C_4, O_4)$.

In the present paper, we study VERTEX COLORING in these classes and in their superclasses defined by forbidden induced subgraphs with at most 4 vertices.

A helpful tool for investigating computational complexity of algorithmic problems on finitely defined classes of graphs is the notion of boundary classes. In Section 2, we use this notion to derive a number of NP-completeness results for VERTEX COLORING. Then in Section 3, we present polynomial-time algorithms for the problem in some classes defined by forbidden induced subgraphs with at most 4 vertices. Together, the results of Sections 2 and 3 cover nearly all classes in the family under consideration. For the remaining classes in this family we prove fixed-parameter tractability of VERTEX COLORING in Section 4. Also, for two of the remaining classes we give a 3/2 approximation polynomial algorithm in Section 5. Section 6 concludes the paper with a number of open problems.

2. Boundary classes and NP-completeness results

As we mentioned in the Introduction, the notion of boundary classes is a helpful tool for the analysis of computational complexity of algorithmic problems on finitely defined classes of graphs. This notion can be defined as follows.

Let Π be an algorithmic graph problem. Let us call a hereditary class $\mathcal{X}$ *Π-easy* if Π can be solved for graphs in $\mathcal{X}$ in polynomial time, and *Π-tough* otherwise. A hereditary class $\mathcal{X}$ is called *Π-limit* if there exists a sequence $\mathcal{X}_1 \supseteq \mathcal{X}_2 \supseteq \dots$ of Π-tough classes such that $\mathcal{X} = \bigcap_{i=1}^{\infty} \mathcal{X}_i$.

To illustrate this notion, let us denote by H_i and L_i the graphs represented in Fig. 1. The following lemma can be found, for instance, in [1].

Lemma 1. *The* MAXIMUM INDEPENDENT SET *problem is NP-complete in the class*

$$Free(C_3, C_4, C_5, \dots, C_p, H_0, H_1, H_2, \dots, H_p)$$

for each fixed value of p.

From this lemma it follows that the class $Free(C_3, C_4, C_5, \dots, H_0, H_1, H_2, \dots)$ is a limit class for the MAXIMUM INDEPENDENT SET problem. Let us denote this class by $\mathcal{S}$. It is not difficult to see that

$\mathcal{S}$ is the class of graphs in which every connected component is of the form $S_{i,j,k}$ represented in Fig. 2(a).