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A sufficient condition to extend polynomial results for the Maximum Independent Set Problem

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ABSTRACT

The Maximum Weight Independent Set Problem (WIS) is a well-known NP-hard problem. A popular way to study WIS is to detect graph classes for which WIS can be solved in polynomial time, with particular reference to hereditary graph classes (i.e., defined by a hereditary graph property), or equivalently to $\mathcal{F}$-free graphs for a given graph family $\mathcal{F}$ (i.e., graphs which are F-free for all $F \in \mathcal{F}$).

A tool to extend the results which show that for given hereditary graph classes the WIS problem can be solved in polynomial time is given by the following easy proposition: For any graph family $\mathcal{F}$, if WIS can be solved for $\mathcal{F}$-free graphs in polynomial time, then WIS can be solved for $K_1 + \mathcal{F}$-free graphs (i.e., graphs which are $K_1 + F$-free for all $F \in \mathcal{F}$) in polynomial time.

The main result of this paper is the following: A sufficient condition to extend the above proposition to $K_2 + \mathcal{F}$-free graphs, and more generally to $lK_2 + \mathcal{F}$-free graphs for any constant l (i.e., graphs which are $lK_2 + F$-free for all $F \in \mathcal{F}$), is that $\mathcal{F}$-free graphs are *m-plausible* for a constant m, i.e., that for any $\mathcal{F}$-free graph G the family of those maximal independent sets I of G such that every vertex of G not in I has more than m neighbors in I can be computed in polynomial time. In this context a section is devoted to show that (for instance) chordal graphs are m-plausible for a constant m.

The proof of the main result is based on the idea/algorithm introduced by Farber to prove that every $2K_2$-free graph has $O(n^2)$ maximal independent sets (Farber, 1989), which directly leads to a polynomial time algorithm to solve WIS for $2K_2$-free graphs through a dynamic programming approach, and on some extensions of that idea/algorithm.
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1. Introduction

An *independent set* (or a *stable set*) of a graph G is a subset of pairwise nonadjacent vertices of G. An independent set of a graph G is *maximal* if it is not properly contained in any other independent set of G.

The *Maximum Weight Independent Set Problem* (*WIS*) is the following: Given a graph G and a weight function w on $V(G)$, determine an independent set of G of maximum weight, where the weight of an independent set I is given by the sum of $w(v)$ for $v \in I$. Let $\alpha_w(G)$ denote the maximum weight of any independent set of G. The WIS problem reduces to the *IS* problem if all vertices v have the same weight $w(v) = 1$.

The WIS problem is NP-hard [16]. It remains NP-hard under various restrictions, such as e.g. triangle-free graphs [29] (and more generally graphs with no induced cycle of given length), cubic graphs [15] and more generally k-regular graphs [13], planar graphs [17], graph classes defined by forbidding a finite set of induced subgraphs having a special structure [3]. It can

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be solved in polynomial time for various graph classes, such as e.g. P_4-free graphs [7] by modular decomposition and more generally perfect graphs [18] by polyhedral combinatorics, claw-free graphs [9,25,27,28,32] (the recent fast algorithm of [9] is based on a decomposition theorem) and more generally fork-free graphs [2,20] by modular decomposition and apple-free graphs [6,4] by decomposition via clique separators, $2K_2$-free graphs [10] by dynamic programming and more generally mK_2-free graphs for any fixed m (by combining an algorithm generating all maximal independent sets of a graph [33] and a polynomial upper bound on the number of maximal independent sets in mK_2-free graphs [1,11,30]), K_2+claw-free graphs [22] by an extension of the approach for $2K_2$-free graphs, and $2P_3$-free graphs [23] similarly. Furthermore the WIS problem can be solved in polynomial time for P_5-free graphs by minimal triangulations, as recently proved in [19].

As one can easily check, for any graph G we have

$$\alpha_w(G) = max\{\alpha_w(G[V(G) \setminus N(v)]) : v \in V\}.$$

Then the WIS problem for any graph G can be reduced to the same problem for the non-neighborhood of all vertices of G. Consequently:

Proposition 1. *For any graph family $\mathcal{F}$, if (W)IS can be solved for $\mathcal{F}$-free graphs in polynomial time, then (W)IS can be solved for $K_1 + \mathcal{F}$-free graphs in polynomial time.* □

The aim of this paper is to try to study in which extent a result similar to Proposition 1 holds once we consider $K_2 + \mathcal{F}$-free graphs instead of $K_1 + \mathcal{F}$-free graphs, that is, in which extent the above considerations hold once we consider the non-neighborhood of edges instead of the non-neighborhood of vertices.

The main result of this paper is the following: A sufficient condition to extend Proposition 1 to $K_2 + \mathcal{F}$-free graphs, and more generally to $lK_2 + \mathcal{F}$-free graphs for any constant l, is that $\mathcal{F}$-free graphs are *m-plausible* for a constant m, i.e., that for any $\mathcal{F}$-free graph G the family of those maximal independent sets I of G such that every vertex of G not in I has more than m neighbors in I can be computed in polynomial time. In this context a section is devoted to show that (for instance) chordal graphs are m-plausible for a constant m.

The proof of the main result is based on the idea/algorithm introduced by Farber to prove that every $2K_2$-free graph has $O(n^2)$ maximal independent sets [10], which directly leads to a polynomial time algorithm to solve WIS for $2K_2$-free graphs through a dynamic programming approach, and on some extensions of that idea/algorithm [21,22].

Basic notation

For any missing notation or reference let us refer the reader to [5].

For any graph G, let $V(G)$ and $E(G)$ denote respectively the vertex-set and the edge-set of G. For any vertex-set $U \subseteq V(G)$, let $N_G(U) = \{v \in V(G) \setminus U : v \text{ is adjacent to some } u \in U\}$ be the *neighborhood of U in G*, and $A_G(U) = V(G) \setminus (U \cup N(U))$ be the *antineighborhood* or *non-neighborhood of U in G*. If $U = \{u_1, \ldots, u_k\}$, then let us simply write $N_G(u_1, \ldots, u_k)$ instead of $N_G(U)$, and $A_G(u_1, \ldots, u_k)$ instead of $A_G(U)$. For any subset $U \subseteq V(G)$ let $G[U]$ be the subgraph of G induced by U. For any vertex $v \in V(G)$ and for any subset $U \subset V(G)$ (with $v \notin U$), let us say that: v*contacts*U if v is adjacent to some vertex of U; v*dominates*U if v is adjacent to each vertex of U. A *component of* G is the vertex set of a maximal connected subgraph of G. A component of G is *trivial* if it is a singleton, and *nontrivial* otherwise. A *clique* of G is a set of pairwise adjacent vertices of G.

A graph G is *H-free* if G contains no induced subgraph isomorphic to a given graph H; in particular, H is called a forbidden induced subgraph of G. Given two graphs G and F, let $G + F$ denote the disjoint union of G and F; in particular, $lG = G + G + \cdots + G$ is the disjoint union of l copies of G.

Given a graph family $\mathcal{F}$, let us say that: a graph is *$\mathcal{F}$-free* if it is F-free for all $F \in \mathcal{F}$; a graph is *$K_1 + \mathcal{F}$-free* if it is $K_1 + F$-free for all $F \in \mathcal{F}$; a graph is *$lK_2 + \mathcal{F}$-free*, for any constant l, if it is $lK_2 + F$-free for all $F \in \mathcal{F}$.

The following specific graphs are mentioned herein. A K_n is a complete graph of n vertices. A P_k has vertices $v_1, v_2, \ldots, v_k$ and edges v_jv_{j+1} for $1 \leq j < k$. A C_k has vertices $v_1, v_2, \ldots, v_k$ and edges v_jv_{j+1} for $1 \leq j < k$ and v_kv_1. A $K_{1,p}$ is the graph formed by an independent set I of p vertices, plus one vertex v which dominates I: a $K_{1,p}$ is also called a *star*, with *center* the vertex v, and with *leaves* the vertices of I. A $Y_{m,m}$ is the graph formed by two disjoint stars $K_{1,m}$ plus one vertex which is adjacent to the centers of such stars. A *claw* has vertices a, b, c, d, and edges ab, ac, ad (then a claw is a $K_{1,3}$). A *fork* has vertices a, b, c, d, e, and edges ab, ac, ad, de (then a fork contains a claw as induced subgraph). A *chordal* graph is a C_k-free graph for all $k \geq 4$.

For a graph G a *vertex-ordering* $(v_1, v_2, \ldots, v_n)$ of G is a total ordering of the vertices of G.

2. Independent sets in $2K_2$-free graphs

Let us report from [22] an algorithm, namely Algorithm Alpha, which formalizes the aforementioned idea/algorithm introduced by Farber [10] and which is the basis of all algorithms presented in the next sections.

The subsequent Algorithm Alpha, for any input $2K_2$-free graph G, produces a family $\mathcal{S}$ of subsets of $V(G)$ each inducing an independent set of G, which can be computed in polynomial time (i.e., $O(n^3)$) and which contains polynomially many members (i.e., $O(n^2)$), such that each maximal independent set of G equals to some member of $\mathcal{S}$.

For a graph G and for a vertex-ordering $(v_1, v_2, \ldots, v_n)$ of G let us denote by G_i the subgraph of G induced by vertices $v_1, v_2, \ldots, v_i$.

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