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Characterizations of Various Continuities of Posets Via Approximated Elements

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Abstract

In this paper, various continuities of posets which may not be dcpos are considered. The concepts of approximated elements and hyper-approximated elements on posets are introduced. New characterizations of continuous posets and hypercontinuous posets are given. Meanwhile, as a generalization of approximated elements, the concept of quasi-approximated elements on dcpos is introduced and some characterizations of quasicontinuous domains are also obtained. It is proved that under some reasonable conditions, the set $\mathrm{B}(L)$ (resp., $\mathrm{QB}(L)$) of approximated elements (resp., quasi-approximated elements) in the induced order of a dcpo L is a continuous domain (resp., a quasicontinuous domain).

Keywords: continuous poset; hypercontinuous poset; approximated element; interpolation property; quasicontinuous domain

1 Introduction

The notion of continuous lattices as a model for the semantics of programming languages was introduced by Scott in [10]. Later, a more general notion of continuous directed complete partially ordered sets (i.e., continuous dcpos or domains) was introduced and extensively studied (see [1], [5], [6]). Since some naturally arising

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posets are important but fail to be directed complete, there are more and more occasions to study posets which fail to possess all directed suprema (see [7]-[9], [11]-[14]). Lawson in [6] gave a remarkable characterization that a dcpo is continuous iff its Scott topology is completely distributive. By the technique of embedded bases and sobrification via the Scott topology, Xu in [11] successfully embedded continuous posets into continuous domains and proved that a poset is continuous iff its Scott topology is completely distributive. Quasicontinuous domains were introduced by Gierz, Lawson and Stralka (see [3]) as a common generalization of both generalized continuous lattices (see [2]) and continuous domains. It was proved that quasicontinuous domains have many properties similar to that of continuous domains and quasicontinuous domains equipped with the Scott topologies are precisely the spectra of distributive hypercontinuous lattices. In terms of intrinsic topologies on posets, Mao and Xu in [8] introduced the concept of hypercontinuous posets and quasicontinuous posets and proved that a poset is quasicontinuous iff its Scott topology is a hypercontinuous lattice.

In [15], Zhao introduced the concept of weakly approximated elements on complete lattices and gave several characterizations of continuous lattices and completely distributive lattices. According to Zhao, an element x of a complete lattice L is said to be weakly approximated if it holds that $x = \bigvee \Downarrow x$. Zhao derived a novel characterization [15, Theorem 3] of continuous lattice that a complete lattice L is continuous iff L satisfies (i) the interpolation property (INT) for the way-below relation $\ll$ on L and (ii) for any $x, y \in L$, $x \neq y$ implies $\Downarrow x \neq \Downarrow y$. He also constructed two counterexamples (see [15, p.163]) to show that none of the conditions (i) and (ii) may be omitted.

In this paper, we introduce the concepts of approximated elements and hyper-approximated elements on posets and discuss some properties and relations of approximated elements and hyper-approximated elements. With these new concepts, we give several characterizations of continuous posets and hypercontinuous posets, generalizing relevant results in [15]. Meanwhile, as a generalization of approximated elements, the concept of quasi-approximated elements on dcpos is also introduced and some new characterizations of quasicontinuous domains are obtained. We will prove that under some reasonable conditions, the set $\mathrm{B}(L)$ (resp., $\mathrm{QB}(L)$) of approximated elements (resp., quasi-approximated elements) in the induced order of a dcpo L is a continuous domain (resp., a quasicontinuous domain).

2 Preliminaries

We quickly recall some basic notions and results (see [1], [3], [8] and [11]). Let $(L, \leq)$ be a poset. A *principal ideal* (resp., *principal filter*) is a set of the form $\downarrow x = \{y \in L \mid y \leqslant x\}$ (resp., $\uparrow x = \{y \in L \mid x \leqslant y\}$). For $A \subseteq L$, we write $\downarrow A = \{y \in L \mid \exists\, x \in A, y \leqslant x\}$, $\uparrow A = \{y \in L \mid \exists\, x \in A, x \leq y\}$. A subset A is a(n) *lower set* (resp., *upper set*) if $A = \downarrow A$ (resp., $A = \uparrow A$). We say that z is a(n) *lower bound* (resp., *upper bound*) of A if $A \subseteq \uparrow z$ (resp., $A \subseteq \downarrow z$). The set of lower bounds of A is denoted by $\mathrm{lb}(A)$. The supremum of A is denoted by $\bigvee A$ or $\sup A$. The