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Countably Sober Spaces

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Abstract

As a generalization of sober spaces, we introduce the concept of countably sober spaces and prove that some topological constructions preserve countable sobriety. In particularly, we prove that the category with countably sober spaces and continuous mappings is a complete category. We give some characterizations of countably sober spaces via countable filters and obtain the Hofmann-Mislove Theorem for countably sober spaces.

Keywords: countably sober space, countable filter, σ-Scott topology, P-space

1 Introduction

Sobriety is between T_0 and T_2 in topological spaces, and being T_1 and being sober are incomparable properties. Sober spaces have wonderful properties and play important roles in domain theory (see, e.g., [1,3,6,8,9,10,11]). For instance, a sober space is a directed complete poset (dcpo, for short) with respect to its *specialization order*, two sober spaces are homeomorphic iff their lattices of open subsets are order isomorphic, and the celebrated Hofmann-Mislove Theorem shows that there is an order isomorphism between the poset of compact saturated subsets and the poset of Scott open filters of open subsets in a sober space. There are many generalizations of sober spaces. In [10], the authors introduced the weaker notion of sobriety, which is called *bounded sobriety*, and proved that the subcategory of bounded sober spaces is reflective in the category of T_0 spaces. D. Zhao and W. K. Ho generalized

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bounded sober spaces to *k-bounded sober spaces* and obtained many interesting results in [11]. In this paper, we intend to generalize sober spaces to *countably sober spaces*.

Recall the definition of *sober spaces*. A subset C of a topological space X is *irreducible* if it is nonempty and if $C \subseteq A \cup B$, where A and B are closed, implies that $C \subseteq A$ or $C \subseteq B$. A topological space X is sober if for every irreducible closed set C, there exists a unique $x \in X$ such that $C = \mathrm{cl}(\{x\})$. Equivalently, a nonempty set C is irreducible if for finite closed set $B_1, B_2, ..., B_n$ in X, $C \subseteq \bigcup_{i=1}^{n} B_i$ implies that $C \subseteq B_i$ for some $i \in \{1, 2, ..., n\}$. Replacing finite closed sets in the definition of irreducible sets by countable closed sets, we define the notion of *countably irreducible sets* and introduce the concept of countably sober spaces via countably irreducible sets.

Countably approximating poset is a successful generalization of continuous poset (see [5,4]). To characterize countably approximating posets, the authors [4] introduced σ-Scott topology on a poset. We show that a countably approximating poset with its σ-Scott topology is a countably sober space, and prove that the topology generated by the σ-Scott topology and the lower topology on a bounded complete and countably directed complete poset is Lindelöf.

We investigate the properties of countably sober space. We prove that some topological constructions preserve countable sobriety. In particularly, we prove that the category with countably sober spaces and continuous mappings is a complete category. We give some characterizations of countably sober spaces via countable filters and obtain the Hofmann-Mislove Theorem for countably sober spaces.

2 Preliminaries

In this section, we recall some basic definitions and notations needed in this paper; more details can be founded in [1,3]. For a set X, the family of all finite sets (resp., countable sets) in X is denoted by Fin X (resp., Count X). For a poset P, $x \in P$, and $A \subseteq P$, let $\downarrow x = \{y \in P : y \leq x\}$, $\downarrow A = \bigcup\{\downarrow x : x \in A\}$; $\uparrow x$ and $\uparrow A$ are defined dually. A subset D of P is called *countably directed* if for any $E \in \mathrm{Count}D$, there exists $d \in D$ such that $E \subseteq \downarrow d$. P is said to be a *countably directed complete poset* if every countably directed subset of P has the least upper bound in P.

For a topological space X, let $\mathcal{O}(X)$ be the lattice of all open subsets in X. For $x \in X$, let $\mathcal{N}(x)$ and $\mathcal{O}(x)$ be the *neighbourhood* and *open neighbourhood* of point x in X, respectively. That is, $\mathcal{N}(x) = \{U \subseteq X :$ there exists $O \in \mathcal{O}(X)$ such that $x \in O \subseteq U\}$, and $\mathcal{O}(x) = \{U \in \mathcal{O}(X) : x \in U\}$. For a T_0 space (X, τ), the *specialization order* $\leq$ on X is defined by $x \leq y$ if and only if $x \in \mathrm{cl}(\{y\})$. Unless otherwise stated, throughout the paper, whenever an order-theoretic concept is mentioned, it is to be interpreted with respect to the specialization order on X.

Definition 2.1 ([5,4]) Let P be a countably directed complete poset. A binary relation $\ll_c$ on P is defined as follows: $x \ll_c y$ iff for any countably directed set $D \subseteq P$, $y \leq \bigvee D$ implies that $x \leq d$ for some $d \in D$. Let $\downarrow_c x = \{y \in P : y \ll_c x\}$. P